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ON UNBOUNDED INVARIANT MEASURES OF STOCHASTIC DYNAMICAL SYSTEMS

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Abstract. We consider stochastic dynamical systems on $\mathbb{R}$, i.e. random processes defined by $X_n^x = \Psi_n(X_{n-1}^x)$, $X_0^x = x$, where Ψ_n are i.i.d. random continuous transformations of some unbounded closed subset of $\mathbb{R}$. We assume here that Ψ_n behaves asymptotically like $A_n x$, for some random positive numbers A_n (the key example being be the affine recursions $\Psi_n(x) = A_n x + B_n$). Our aim is to describe invariant Radon measures of the process X_n^x, in the critical case, when $\mathbb{E}\log A_1 = 0$. We prove that those measures behave at infinity like $\frac{dx}{x}$. We study also the problem of uniqueness of the invariant measure. We improve previous results known for the affine recursions and generalize them to a larger class of stochastic dynamical systems that includes, for instance, reflected random walks, stochastic dynamical systems on the unit interval $[0,1]$, additive Markov processes and a variant of the Galton-Watson process.

1. Introduction

Let $\mathfrak{F}$ be the semigroup of continuous transformations of an unbounded closed subset $\mathcal{R}$ of the real line $\mathbb{R}$ endowed with the topology of uniform convergence on compact sets. In the most interesting examples $\mathcal{R}$ is the real line, the half-line $[0,+\infty)$ or the set of natural numbers $\mathbb{N}$. Given a regular measure μ on $\mathfrak{F}$, we define the stochastic dynamical system (SDS) on $\mathcal{R}$ by

$$\begin{aligned} X_0^x &= x; \\ X_n^x &= \Psi_n(X_{n-1}^x), \end{aligned} \tag{1.1}$$

where $\{\Psi_n\}$ is a sequence of i.i.d. random functions, distributed according to μ.

The aim of this paper is to study conditions for the existence and uniqueness, as well as behavior at infinities, of an invariant infinite Radon measure of the process X_n^x, i.e. of a measure ν on $\mathbb{R}$ such that

$$\mu *_{\mathfrak{F}} \nu(f) = \nu(f), \tag{1.2}$$

for any $f \in C_C(\mathbb{R})$, where

$$\mu *_{\mathfrak{F}} \nu(f) = \int_{\mathbb{R}} \mathbb{E}\big[f(X_1^x)\big]\nu(dx) = \int_{\mathfrak{F}}\int_{\mathbb{R}} f(\Psi(x))\nu(dx)\mu(d\Psi).$$

There exists quite an extensive literature on the case when the process X_n is positive recurrent, that is possesses an invariant *probability* measure. The existence of such a measure can be proved supposing that the process has some contractive property (for example, if Ψ_n are Lipschitz mappings with Lipschitz coefficients $L_n = L(\Psi_n)$ and $\mathbb{E}[\log L_1] < 0$), [11]). This invariant probability measure is well described in several specific cases, such as affine recursions (i.e. $\Psi(x) = Ax + B$), namely in the seminal paper of H.Kesten [18]. C.M.Goldie [15] and, more recently and more generally, M.

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Mirek [21] have generalized Kesten's theorem to recursions that behave like Ax for large x, and proved that if $\mathbb{E}A^\kappa = 1$ (and some other hypotheses are satisfied), then

$$\lim_{z\to\infty} z^\kappa \nu\{x : |x| > z\} = C_+ > 0.$$

In other words the measure ν behaves at infinity as $\frac{dx}{x^{1+\kappa}}$

Less is known for the null recurrent case, especially in a general setting. Existence and uniqueness of an invariant Radon measure have been the object of two recent works: B. Deroin, V. Kleptsyn, A. Navas and K. Parwani [10], that studies symmetric SDS of homeomorphism of $\mathbb{R}$, and M. Peigné and W. Woess [22] on the phenomenon of local contraction. We refer to them for a more complete bibliography on the subject. As in the contracting case, affine recursions is one of the first model to be systematically approached. A seminal paper in this area is Babillot, Bougerol and Elie [2], where they prove existence and uniqueness of a Radon measure and gave a first result on the behavior at infinity.

The goal of the present work is twofold. First of all we investigate the behavior at infinity of invariant measures and, for a large class of SDS's, we generalize and improves results known for affine recursions. Secondly, we consider the problem of uniqueness of the invariant measure. This problem was studied in the above-mentioned papers, nevertheless all the result stated there are quite general or difficult to use in practise. We give a relatively simple criterium that can be applied for very concrete examples.

1.1. **Behavior at infinity.** It turns out that to prove existence and to describe the tail of the measure it is sufficient to control the maps that generate the SDS near infinity. In particular we suppose that they are asymptotically linear, in the sense that there exists $0 \le \alpha < 1$ such that for all $\psi \in \mathfrak{F}$

$$(AL^\alpha) \qquad |\psi(x) - A_\alpha(\psi)x| \le B_\alpha(\psi)(1 + |x|^\alpha) \qquad \text{for all } x \in \mathcal{R}$$

with $A_\alpha(\psi)$ and $B_\alpha(\psi)$ strictly positive. We study here the critical case, i.e. $\mathbb{E}[\log A] = 0$.

Existence of an invariant measure supported in $\mathcal{R}$ is relatively easy to deduce from well-known literature, because the SDS is in this case bounded by a recurrent process (we give more details in Subsection 2.3). The main result of the paper is the description of the tail of invariant measures at infinity.

Theorem 1.3. *Suppose that there exists $0 \le \alpha < 1$ such that the maps Ψ_n satisfy (AL^α) μ-almost surely and that*

$$\mathbb{E}[\log A_\alpha] = 0 \text{ and } \mathbb{P}[A_\alpha = 1] < 1, \tag{1.4}$$

$$\mathbb{E}\big[(|\log A_\alpha| + \log^+ |B_\alpha|)^{2+\varepsilon}\big] < \infty \tag{1.5}$$

(1.6) *the law of $\log A_\alpha$ is aperiodic, i.e. there is no $p \in \mathbb{R}$ such that $\log A_\alpha \in p\mathbb{Z}$ a.s.*

Then for every invariant Radon measure ν supported by $\mathcal{R}$ of the process $\{X_n^x\}$ that is not null at a neighborhood of $+\infty$, the family of dilated measures $\delta_{z^{-1}} * \nu(I) := \nu(zI)$ converges weakly on $\mathbb{R}_+^*$ to $C_+\frac{da}{a}$ as z goes to infinity for some $C_+ > 0$, i.e.

$$\lim_{z\to\infty} \int_{\mathbb{R}_+^*} \phi(z^{-1}u)\nu(du) = C_+ \int_{\mathbb{R}_+^*} \phi(a)\frac{da}{a},$$

for any $\phi \in C_C(\mathbb{R}_+^*)$.

The key example of asymptotic linear SDS is the affine recursions (called also the random difference equation), i.e. $\mathfrak{F}$ consisting of affine mappings Ψ of the real line: $\Psi(x) = Ax + B$ for some $A \ge 0$. Then

$$X_n^x = A_n X_{n-1}^x + B_n, \qquad X_0^x = x. \tag{1.7}$$

Our results are also valid for Goldie's recursions (e.g. $\Psi(x) = \max\{Ax, B\} + C$ (with $A > 0$) and $\Psi(x) = \sqrt{A^2x^2 + Bx + C}$ (with A, B, C positive; for other condition needed to have uniqueness, see section 6) . Since the problem can be reduced, without any loss of generality, to the case $\alpha = 0$ (see Lemma 2.1) , our hypothesis essentially coincides, in the one dimensional situation, with the class introduced by M.Mirek [21]. Our main theorem should be viewed as an analog of Kesten's and Goldie's results in the critical case.

Other interesting examples can be obtained conjugating asymptotic linear systems with an appropriate homeomorphism. For instance, our result can also be applied to describe invariant measures of SDS on the interval generated by functions that have the same derivative at the two extremities. Theorem 1.3 also says that invariant measures of SDS on $[0, +\infty)$ generated by mappings exponentially asymptotic to translations, i.e.

$$|\psi(x) - x + u_\psi| \leq v_\psi \mathrm{e}^{-x}, \qquad \forall x \geq 0$$

behave at infinity as the Lebesgue measure dx of $\mathbb{R}$, if $\mathbb{E}(u_\phi) = 0$. This result can be compared with the Choquet-Deny Theorem, that says that the only invariant measure for centered random walks on $\mathbb{R}$ is the Lebesgue measure. Another interesting process that is α-asymptotically linear for $\alpha > 1/2$ is a Galton-Watson evolution process with random reproduction laws. In Section 6, we give more details on the different examples.

Let us mention that in our previous papers [3, 6, 7] we have already studied the behavior of the invariant measure ν at infinity for the random difference equation (1.7). However the main results were obtained there under much stronger assumptions, namely we assumed existence of exponential moments, that is $\mathbb{E}[A^\delta + A^{-\delta} + |B|^\delta] < \infty$ for some $\delta > 0$. Theorem 1.3 improves all our previous results for affine recursions and describes the asymptotic behavior of ν under minimal assumptions, that the hypotheses implying existence of the invariant measure. Up to our knowledge, for all the other recursions even partial results were not known.

We would like also to remark that, in the contracting case, Kesten's theorem require moment of order at least κ and, as far as we know, there exits no result on the behavior of the tail of the invariant probability when the measure is supposed to have only logarithmic moment.

The proof of Theorem 1.3 is given in sections 3 and 4. In order to describe ν at infinity, we give first an upper bound of this measure and prove some regularity properties of its quotient. The techniques we use in the present paper to prove this facts are more powerful than those we used in [6], and are heavily based on the renewal theory for random walks on the affine group. Among other results we prove directly that $\nu[-z, z]$ grows as $\log z$ (Proposition 3.1). Next, in Section 4 we consider the Poisson equation on $\mathbb{R}$

$$f(x) = \overline{\mu} * f(x) + g(x),$$

where $f(x) = \int \phi(e^{-x}u)\nu(du)$ for some $\phi \in C_C(\mathbb{R}_+^*)$ and $\overline{\mu}$ is the law of $-\log A$. Notice the the asymptotic behavior of f and ν is the same, thus it is sufficient to study f. In the contrast to [6] we do not solve explicitly this equation. We apply techniques borrowed from the work of Durrett and Liggett [12] (see also Kolesko [19]) and reduce the problem to consider f as a solution of the classical renewal equation and deduce its asymptotic behavior from the renewal theorem.

1.2. **Uniqueness of the invariant measure.** An other fundamental question is to determine whether the invariant measure is unique or not. The nature of this problem is different from the ones we have considered so far, since it is a more local property. Thus it is not possible to obtain some results only by controlling the random maps at the infinity.

In the non-contracting case, this problem was studied first by Babillot, Bougerol and Elie [2] in the context of the affine recursion and they proved uniqueness under the assumptions of Theorem 1.3.

Relying on their ideas Benda [3] studied in full generality recurrent and locally contractive SDS's. The SDS is called recurrent if there exists a closed set L such that every open set intersecting L is visited by X_n^x infinitely often with probability 1. The SDS is locally contractive if for any $x, y \in \mathbb{R}$ and every compact set $K \subset \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\big[|X_n^x - X_n^y| \cdot \mathbf{1}_K(X_n^x)\big] = 0. \tag{1.8}$$

Benda [3] proved that if $\{X_n^x\}$ is a recurrent and locally contractive SDS, then it possesses a unique (up to a multiplicative constant) invariant Radon measure. He didn't publish his results, however they have been recently incorporated, with a complete and simplified proof, into two papers of Peigné and Woess [22, 23], where they also investigated ergodicity of SDS generated by Lipschitz maps with centered Lipschitz's coefficient. A different approach can be found in [10], where the authors prove a local contraction properties for a symmetric SDS generated by homeomorphisms of $\mathbb{R}$.

Our contribution to the subject is to prove a sufficient condition for uniqueness that can be applied to some concrete mappings of $\mathbb{R}_+$ that have a certain interest in applications, such as such $\psi(x) = \max\{Ax, B\} + C$ and most of Goldie's recursions.

Theorem 1.9. *Suppose that $\mathcal{R} = [0, \infty)$, $\alpha = 0$ and that hypotheses of Theorem 1.3 are satisfied. Assume moreover that*

(1) There exists $\beta > 0$ such that $\mathbb{P}(\Psi[0, +\infty) \subseteq [\beta, +\infty)) > 0$
(2) $A(\Psi)x \le \Psi(x) \le A(\Psi)x + B(\Psi)$ for all $x \ge 0$
(3) The functions Ψ are Lipschitz and their Lipschitz coefficients are equal to $A(\Psi)$.

Then the SDS defined on $[0, \infty)$ by (1.1) is locally contractive. Therefore there exists a unique invariant Radon measure of the process $\{X_n^x\}$ on $[0, +\infty)$.

1.3. **Reflected random walk.** The reflected random walk is the SDS defined for $x \in \mathbb{R}^+ = [0, \infty)$, by

$$\begin{aligned} Y_0^x &= x, \\ Y_n^x &= \big|Y_{n-1}^x - u_n\big|, \end{aligned} \tag{1.10}$$

where u_n is a sequence of i.i.d. real valued random variables with a given law μ.

If $u_n \ge 0$ a.s., then it was proved by Feller [14] that this process possesses a unique invariant probability measure ν, i.e. a measure satisfying

$$\mu * \nu(f) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(|x-y|)\nu(dx)\mu(dy) = \int_{\mathbb{R}_+} f(x)\nu(dx) = \nu(f).$$

Moreover the measure ν can be explicitly computed: $\nu(dx) = (1 - F(x))dx$, for F being the distribution function of μ. The process has been also studied in more general settings, when u_n admits also negative values (see Peigne, Woess [22] for recent results and a comprehensive bibliography).

Here, we are interested in the critical case when $\mathbb{E}u_n = 0$. Peigne and Woess [22] proved that if $\mathbb{E}(u_1^+)^{\frac{3}{2}} < \infty$, for $u_1^+ = \max\{u_1, 0\}$, then the process $\{X_n\}$ is recurrent on $\mathbb{R}_+$. As a consequence, by Benda's theorem, it possesses a unique invariant Radon measure ν on $\mathbb{R}_+$ (local contractivity is easy to prove). The reflected random walk can be transformed in an asymptotically linear system by conjugating with an invertible function s of $[0, +\infty)$ such that $s(x) = \mathrm{e}^x$ for large x. Then $\psi(x) = s(|s^{-1}(x) - u|)$ is asymptotically linear with $A(\psi) = e^{-u}$. Thus Theorem 1.3 can be used to to prove that the invariant measure of Y_n^x behaves at infinity like the Lebesgue measure. Nevertheless in this case one can prove the same result under weaker moment assumptions and a much simpler proof. Using a simple argument based only on the duality lemma and the renewal theorem gives:

Theorem 1.11. *Assume $\mathbb{E}u_1 = 0$, $\mathbb{E}(u_1^+)^{\frac{3}{2}} < \infty$, $\mathbb{E}(u_1^-)^2 < \infty$ and the law μ of u_1 is aperiodic, then for every $\phi \in C_C(\mathbb{R}_+)$*

$$\lim_{x\to\infty} \int_{\mathbb{R}_+} \phi(u-x)\nu(du) = C_+ \int_{\mathbb{R}_+} \phi(u)du,$$

for some positive constant C_+.

Proof of this theorem will be given in Section 6.6

2. Notations and preliminary results

2.1. **Reduction to condition (AL).** First of all we would like to observe that, conjugating the SDS with an appropriate function, we can suppose without loss of generality that the error to linearity is constant. In fact we have the following lemma whose proof is postponed to Appendix A:

Lemma 2.1. *Let $0 \le \alpha < 1$ suppose that ψ satisfies*

$$(AL^\alpha) \qquad |\psi(x) - A_\alpha x| \le B_\alpha(1 + |x|^\alpha)$$

then if $r(x) = \mathrm{sign}(x)|x|^{1-\alpha}$ then the conjugate function $\psi_r = r \circ \psi \circ r^{-1}$ satisfies (AL^0) with $A_0 = A_\alpha^{1-\alpha}$ and B_0 can be chosen such that $\log^+ B_0 \le C_\alpha(|\log A_\alpha| + \log^+ B_\alpha + 1)$, where the constant C_α depends only on α.

Then if ψ is distributed according to μ, the law ψ_r is given by $\mu_r = r * \mu * r^{-1}$, and if ν is a μ-invariant measure then $\nu_r = r * \nu$ is μ_r-invariant. Thus if Theorem 1.3 holds for ν_r then it holds for ν in fact

$$\begin{aligned}
\lim_{z\to\infty} \int_{\mathbb{R}_+^*} \phi(z^{-1}u)\nu(du) &= \lim_{z\to\infty} \int_{\mathbb{R}_+^*} \phi(z^{-1}r^{-1}(u))\nu_r(du) = \lim_{z\to\infty} \int_{\mathbb{R}_+^*} \phi(z^{-1}u^{1/(1-\alpha)})\nu_r(du) \\
&= \lim_{z\to\infty} \int_{\mathbb{R}_+^*} \phi((z^{-(1-\alpha)}u)^{1/(1-\alpha)})\nu_r(du) = C_+ \int_{\mathbb{R}_+^*} \phi(a^{1/(1-\alpha)})\frac{da}{a} \\
&= C_+(1-\alpha) \int_{\mathbb{R}_+^*} \phi(a)\frac{da}{a}.
\end{aligned}$$

For this reason in order to avoid useless notations, we'll suppose from now on that $\alpha = 0$, that is we suppose that for all $\psi \in \mathfrak{F}$

$$(AL) \qquad A(\psi)x - B(\psi) < \psi(x) < A(\psi)x + B(\psi) \qquad \text{for all } x \in \mathcal{R}$$

Since we supposed that $\mathcal{R}$ is closed, we can extend the property (AL) to all $x \in \mathbb{R}$ for a suitable continuous extension of ψ to $\mathbb{R}$. With a slight abuse of notation, we will denote with the same letter (e.g. ψ), the map from $\mathcal{R}$ to $\mathcal{R}$ and a continuous extension that verify (AL) for all $x \in \mathbb{R}$. At the same way, ν will be seen both as a measure on $\mathcal{R}$ and as a measure on $\mathbb{R}$ whose support is contained in $\mathcal{R}$.

We assume that the maps $A = A(\psi)$ and $B = B(\psi)$ from $\mathfrak{F}$ to $\mathbb{R}_+^* = (0, \infty)$ are measurable and that $\mathfrak{F}$ is a monoid, that is closed by composition. Since (AL) implies that

$$\lim_{\substack{x\to+\infty \\ x\in\mathcal{R}}} \psi(x)/x = \lim_{\substack{x\to-\infty \\ x\in\mathcal{R}}} \psi(x)/x = A(\psi)$$

then the map A is a morphism from $\mathfrak{F}$ to $\mathbb{R}_+^*$ i.e. $A(\psi_1 \circ \psi_2) = A(\psi_1)A(\psi_2)$. The choice of B is not unique and can be chosen as big as needed.

2.2. **Comparison of X_n^x with the affine recursion.** Let Ψ_i be an i.i.d. sequence of random variables with values in $\mathfrak{F}$ of law μ. We are interested in the study of the iterated stochastic function system

$$X_n^x = \Psi_n(X_{n-1}^x) = \Psi_n ... \Psi_1(x) \text{ and } X_0^x = x.$$

We will always assume that hypothesis (AL) is satisfied and thus the trajectories of the process X_n^x can be dominated from below and from above by affine recursions

$$Z_n^x = A_n Z_{n-1}^x - B_n \text{ and } Y_n^x = A_n Y_{n-1}^x + B_n, \tag{2.2}$$

where, to simplify our notation, we note $A_i = A(\Psi_i)$ and $B_i = B(\Psi_i)$. We will also always assume, according to hypotheses of Theorem 1.3, a logarithmic moment of order $2+\varepsilon$ and that $\log A_1$ is nontrivially centered. Without any loss of generality, we can also chose $B(\psi)$, such that:

$$B_i \geq 1 \qquad \text{almost surely} \tag{2.3}$$

$$\frac{B_i}{1-A_i} \qquad \text{is not a.s. constant i.e. } \mathbb{P}(A_i x + B_i = x) < 1 \text{ for all } x. \tag{2.4}$$

In such a way the two dimensional process process (Z_n^x, Y_n^x) satisfies all the assumptions required by Babillot, Bougerol, Elie [2]. Thus it is recurrent, locally contractive and possesses a unique invariant measure.

It will be convenient to use in the proof the language of groups. Namely, let $G = \mathrm{Aff}(\mathbb{R}) = \mathbb{R} \rtimes \mathbb{R}_+^*$ be the group of all affine mappings of $\mathbb{R}$, i.e. the set of pairs $(b,a) \in \mathbb{R} \times \mathbb{R}_+^*$ acting on $\mathbb{R}$: $(b,a): x \mapsto ax + b$. Then the group product is given by the formula

$$(b,a) \cdot (b',a') = (b + ab', aa'),$$

the identity element is $(0,1)$ and the inverse element is given by

$$(b,a)^{-1} = (-b/a, 1/a).$$

Let μ_G be a probability measure on the group G being the law of (B_i, A_i). Then the random elements $g_i = (B_i, A_i)$ are i.i.d. random variables in G with law μ_G. On G we define the left and the right random walk:

$$L_n = g_n \cdot ... \cdot g_1, \qquad R_n = g_1 \cdot ... \cdot g_n. \tag{2.5}$$

Then, $Y_n^x = L_n(x)$.

A very important role in our proofs will be played by the random walk on $\mathbb{R}$ generated by $-\log A_i$, i.e.

$$S_n = -(\log A_1 + \cdots + \log A_n), \tag{2.6}$$

(we put the sign minus to follow notations of previous works). Since $\mathbb{E} \log A = 0$, the random walk S_n is recurrent. Moreover since we assume aperiodicity, the support of S_n is just $\mathbb{R}$. We often use the downward and upward sequence of stopping times

$$l_n := \inf\{k > l_{n-1} : S_k < S_{l_{n-1}}\}, \quad t_n := \inf\{k > t_{n-1} : S_k \geq S_{t_{n-1}}\} \tag{2.7}$$

and $l_0 = t_0 = 0$. Observe that t_1 and l_1 are almost surely finite, but have infinite mean. On the other hand, hypothesis $\mathbb{E}(|\log A|^{2+\varepsilon}) < \infty$ guarantees that S_{t_1} and S_{l_1} are integrable (see [9]).

In the sequel we will use, depending on the situation, different convolutions. We define a convolution of a function f on $\mathbb{R}$ with a measure η on $\mathbb{R}$ as a measure on $\mathbb{R}$ given by

$$f * \eta(K) = \int_{\mathbb{R}} \mathbf{1}_K(f(u)) \eta(du) = \eta(f^{-1}(K)). \tag{2.8}$$

Given $z \in \mathbb{R}_+^*$ and a measure η on $\mathbb{R}$ we define

$$\delta_z *_{\mathbb{R}_+^*} \eta(K) = \int_{\mathbb{R}} \mathbf{1}_K(zu)\eta(du) = \eta(z^{-1}K). \tag{2.9}$$

2.3. **Existence of an invariant measure.** We conclude this section observing that the existence of the invariant measure on $\mathcal{R} \subseteq \mathbb{R}$ for a SDS satisfying hypotheses of Theorem 1.3 follows immediately from recurrence of the process $\{X_n^x\}$ and Lin's theorem [20].

More precisely, consider the stochastic positive operator $Pf(x) = \int f(\Psi(x))\mu(d\Psi)$ on $C_b(\mathcal{R})$. Then since $Z_n^x \leq X_n^x \leq Y_n^x$ and (Z_n^x, Y_n^x) is recurrent, the process $\{X_n^x\}$ is recurrent, i.e. there exists $u \in C_c(\mathbb{R})$ such that $\sum_{n=0}^\infty P^n u(x) = \infty$ for all x. Thus by [20] there exists a non-null invariant Radon measure ν on $\mathcal{R}$ of the process $\{X_n^x\}$.

Observe that the support of this measure can be bounded (for instance if the functions Ψ fix the point 0, then the Dirac measure at 0 is an invariant measure). In this paper we are interested in the case when the measure has infinite mass, thus its support is not compact. A sufficient (but not necessary condition) to have an infinite Radon measure is that the function Ψ do not fix any compact subset C of $\mathbb{R}$ that is there is no C such that $\mathbb{P}(\Psi(C) \subseteq C) = 1$.

3. First bounds of the tail of the invariant measure

We start to study the behavior of ν at infinity. In particular we will prove in this section that $\nu(dx)$ does not grow faster than $\frac{dx}{x}$ the Haar measure of $\mathbb{R}_+^*$. The behavior of ν at ∞ is related to the behavior of the family of measures $\delta_{z^{-1}} * \nu$. In this section we prove

Proposition 3.1. *Under hypotheses of Theorem 1.3:*

(1) There exists $C_0 > 0$ such that

$$\nu[-z, z] < C_0(1 + \log z) \quad \text{for all } z > 1$$

Moreover, if the support of ν is not bounded on the right, i.e. $\nu(z, +\infty) > 0$ for all $z \in \mathbb{R}$, then

(2) There exists M and $\delta > 0$ such that $\nu[z, zM] > \delta$ for all $z \geq 1$

(3) For all $u_2 > u_1 > 0$ there exists $C = C(u_1, u_2, M) > 0$ such that

$$\frac{\nu[\mathrm{e}^{x+y}u_1, \mathrm{e}^{x+y}u_2]}{\nu[\mathrm{e}^x, \mathrm{e}^x M]} < C(1 + y) \quad \text{for all } x > 0, y > 0. \tag{3.2}$$

In particular the family of measures $\frac{\delta_{z^{-1}} * \nu}{\nu[z, zM]}$ on $(0, +\infty)$ is weakly compact when z goes to $+\infty$.

There are two keys arguments in the proof of this proposition. One is the following Lemma, that we will use several time in the sequel.

Lemma 3.3. *Let ν be a positive μ-invariant measure on $\mathbb{R}$, then for any pair of intervals $V, U \subset \mathbb{R}$,*

$$\nu(V) \geq \mathbb{P}(T_{\mathfrak{W}} < \infty) \cdot \nu(U)$$

where

$$\mathfrak{W} = \mathfrak{W}(V, U) = \{\psi \in \mathfrak{F} | \psi(U) \subset V\}$$

and $T_{\mathfrak{W}}$ is the stopping time defined by $T_{\mathfrak{W}} = \inf\{n \geq 0 : \Psi_1 \cdots \Psi_n \in \mathfrak{W}\}$.

Proof. Observe that the backward process

$$M_n = \Psi_1 \cdots \Psi_n * \nu(V) \qquad M_0 = \nu(V)$$

is a positive martingale with respect to the filtration generated by the Ψ_n. In fact

$$\mathbb{E}(M_n | \mathcal{F}_{n-1}) = \Psi_1 \cdots \Psi_{n-1} * \mu * \nu(V) = \Psi_1 \cdots \Psi_{n-1} * \nu(V).$$

Since $(\Psi_1 \cdots \Psi_{T_{\mathfrak{W}}})^{-1}(V) \supseteq U$, for any fixed $n \in \mathbb{N}$, by the optional stopping time theorem,

$$\nu(V) = \mathbb{E}(M_{T_{\mathfrak{W}} \wedge n}) \geq \mathbb{E}(\mathbf{1}_{\{T_{\mathfrak{W}} \leq n\}} \Psi_1 \cdots \Psi_{T_{\mathfrak{W}}} * \nu(V)) \geq \mathbb{P}(T_{\mathfrak{W}} < n)\nu(U).$$

We let n go to infinity to conclude. □

The other crucial observation is that the backward recursion $\Psi_1 \cdots \Psi_n(x)$ is controlled by the right random walk R_n on the affine group generated by the product of $g_i = (B_i, A_i)$ (see (2.5)). More precisely if $g \in \mathrm{Aff}(\mathbb{R})$ denote by $a(g)$ and $b(g)$ the projection on $\mathbb{R}_+^*$ and $\mathbb{R}$ respectively, then

$$a(R_n)x - b(R_n) \leq \Psi_1 \cdots \Psi_n(x) \leq a(R_n)x + b(R_n).$$

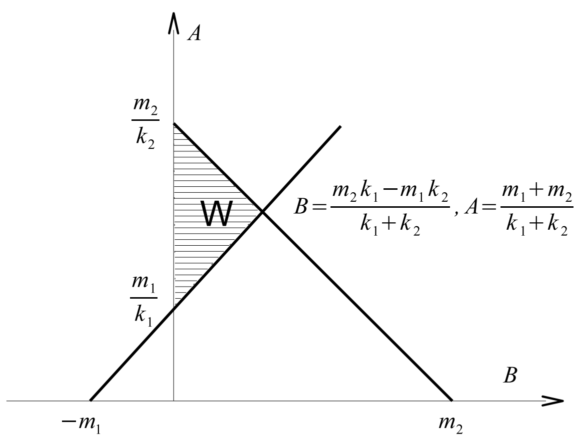
We use these bounds to estimate the stopping time that appears in Lemma 3.3. In particular as an immediate consequence of the lemma above we obtain

Corollary 3.4. *Let*

$$W = W(m_1, m_2, k_1, k_2) = \{(B, A) \in \mathrm{Aff}(\mathbb{R}) | Ak_2 + B \leq m_2; \quad Ak_1 - B \geq m_1\}$$

and $T_W = \inf\{n \geq 0 : R_n \in W\}$. *Then we have*

$$\nu(m_1, m_2) \geq \mathbb{P}[T_W < \infty]\nu(k_1, k_2).$$



Proof. The Corollary follows from Lemma 3.3, taking $U = [k_1, k_2]$, $V = [m_1, m_2]$ and noticing that $T_W \geq T_{\mathfrak{W}}$. □

Since the potential theory of the affine group is well understood, we have enough tools to estimate $\mathbb{P}(T_W < +\infty)$ in many situations. For continuous and compactly supported function f on $\mathrm{Aff}(\mathbb{R})$ we define the potential

$$U * \delta_g(f) := \mathbb{E}\Big[\sum_{n=0}^{\infty} f(L_n g)\Big] = \mathbb{E}\Big[\sum_{n=0}^{\infty} f(R_n g)\Big].$$

A renewal theorem for the potential U, i.e. description of its behavior at infinity, was given in [2], where the authors proved that for all $h \in C_C(\mathrm{Aff}(\mathbb{R}))$:

$$\lim_{a \to 0} U * \delta_{(0,a)}(h) = \nu_G \otimes \frac{dx}{x}(h), \tag{3.5}$$

where ν_G is a suitable non-trivial multiple of the invariant measure of the process $Y_n^x = L_n(x)$.

Now we are ready to prove the following lemma.

Lemma 3.6. *Suppose (1.4), (1.5), (2.3) and (2.4). There exist a compact subset*

$$V_0 = \{(B,A) \in \mathrm{Aff}(\mathbb{R}) | |B| < b_0, a_0^{-1} < A < a_0\}$$

and a constant $\delta > 0$ such that:

(1) *if $W_z = (0,z) \cdot V_0 = \{(B,A) | |B| < z\, b_0,\ z\, a_0^{-1} < A < z\, a_0\}$, then:*

$$\mathbb{P}(T_{W_z} < \infty) > \delta$$

for all $z \geq 1$;

(2) *if $V_z = V_0 \cdot (0, z^{-1}) = \{(B,A) | |B| < b_0,\ a_0^{-1}/z < A < a_0/z\}$, then:*

$$\mathbb{P}(T_{V_z} < \infty) > \frac{\delta}{1 + \log z}$$

for all $z \geq 1$.

Proof. Step 1. First observe that for every $V \subset \mathrm{Aff}(\mathbb{R})$

$$U(V^{-1}V)\mathbb{P}(T_V < \infty) \geq U(V). \tag{3.7}$$

In fact

$$U(V) = \sum_{n=0}^{\infty} \mathbb{P}[R_n \in V] = \mathbb{E}\Big[\mathbf{1}_{\{T_V < \infty\}} \sum_{n=T_V}^{\infty} \mathbf{1}_{\{R_{T_V} R_n^{T_V} \in V\}}\Big] \leq \mathbb{P}(T_V < \infty) U(V^{-1}V)$$

where $R_n^l := R_l^{-1} R_n = g_{l+1} \cdots g_n$.

Step 2: Proof of (1). By (3.7) we write (assuming the denominator is nonzero)

$$\mathbb{P}(T_{W_z} < \infty) \geq \frac{U(W_z)}{U(W_z^{-1} W_z)} = \frac{U((0,z) \cdot V_0)}{U(V_0^{-1} V_0)}. \tag{3.8}$$

A simple calculation relate the right random walk on the affine group with the reversed left random walk $\breve{L}_n = R_n^{-1} = g_n^{-1} \cdots g_1^{-1}$. Observe that for any $V \subset \mathrm{Aff}(\mathbb{R})$ we have

$$\begin{aligned} U((0,z)V) &= \sum_n \mathbb{P}\big[R_n \in (0,z)V\big] = \sum_n \mathbb{P}\big[R_n^{-1} \in V^{-1}(0,z^{-1})\big] \\ &= \sum_n \mathbb{P}\big[\breve{L}_n(0,z) \in V^{-1}\big] = \breve{U}\big(V^{-1}(0,z^{-1})\big), \end{aligned}$$

where $\breve{U}$ is the potential related to the reversed random walk $\breve{L}_n$. Since the law of g_n^{-1} is also centered and verifies hypotheses of [2], there exists a unique Radon measure $\breve{\nu}_G$ on $\mathbb{R}$ invariant under $\breve{\mu}_G$, the law of $g^{-1} = (B,A)^{-1}$. Then by (3.5)

$$\lim_{z \to +\infty} U((0,z)V) = \lim_{z \to +\infty} \breve{U}\big(V^{-1}(0,z^{-1})\big) = \Big(\breve{\nu}_G \times \frac{dx}{x}\Big)(V^{-1}).$$

We take V_0 big enough such that

$$U(W_z^{-1} W_z) = U(V_0^{-1} V_0) > 0 \quad \text{and} \quad \Big(\breve{\nu}_G \times \frac{dx}{x}(V_0^{-1})\Big) > 0$$

and, in view of (3.7), we conclude.

Step 3: Proof of (2). As in the previous step, by (3.7), we write

$$\mathbb{P}(T_{V_z} < \infty) > \frac{U(V_z)}{U(V_z^{-1} V_z)} = \frac{U\big(V_0(0,z^{-1})\big)}{U\big((0,z)V_0^{-1}V_0(0,z^{-1})\big)} \tag{3.9}$$

and we have to estimate $U(V_z)$ from below and $U(V_z^{-1}V_z)$ from above, that is the most difficult part of the proof.

Let $\{\overline{g}_i\}$ be another sequence of i.i.d. elements of $\mathrm{Aff}(\mathbb{R})$ independent and of the same law as $\{g_i\}$. We define $\overline{S}_n, \overline{t}_k, \overline{l}_k$ as in (2.6) and (2.7). We first claim

$$U(f) = \mathbb{E}\Big[\sum_{n=0}^{\infty} f(L_n)\Big] = \mathbb{E}\Big[\sum_{k,i=0}^{\infty} f(\overline{R}_{\overline{l}_i} L_{t_k})\Big] \tag{3.10}$$

In fact, for $n > k$ define $L_n^k = g_n \cdots g_{k+1}$ and $L_k^k = e$. Observe that

$$\mathbb{E}\Big[\sum_{n=0}^{\infty} f(L_n)\Big] = \mathbb{E}\Big[\sum_{k=0}^{\infty}\sum_{i=t_k}^{t_{k+1}-1} f(L_i)\Big] = \mathbb{E}\Big[\sum_{k=0}^{\infty}\mathbb{E}\Big[\sum_{i=t_k}^{t_{k+1}-1} f(L_i^{t_k} L_{t_k})\Big| L_{t_k}\Big]\Big].$$

Since for fixed k the sequence $\{L_{t_k+i}^{t_k}\}_{i\geq 0}$ is independent from L_{t_k} and has the same law as $\{L_i\}_{i\geq 0}$, by the duality lemma (see lemma 5.4 [6]) we have

$$\mathbb{E}\Big[\sum_{i=t_k}^{t_{k+1}-1} f(L_i^{t_k} L_{t_k}) | L_{t_k} = g\Big] = \mathbb{E}\Big[\sum_{i=0}^{t_1-1} f(L_i g)\Big] = \mathbb{E}\Big[\sum_{i=0}^{\infty} f(\overline{R}_{\overline{l}_i} g)\Big]$$

and we obtain (3.10).

Observe that $\overline{S}_{\overline{l}_i}$ (resp. S_{t_k}) is a random walk of finite mean and negative (resp. positive) steps. Take $a, b > 2$, then by (3.10) and the classical renewal theorem [14], we have

$$\begin{aligned}
U([-b,b]\times[1/a,a]) &= \sum_{k,i=0}^{\infty} \mathbb{P}\big[b(\overline{R}_{\overline{l}_i} L_{t_k}) \leq b; -\log a \leq \overline{S}_{\overline{l}_i} + S_{t_k} \leq \log a\big] \\
&= \sum_{k,i=0}^{\infty} \mathbb{P}\big[\mathrm{e}^{-\overline{S}_{\overline{l}_i}} b(L_{t_k}) + b(\overline{R}_{\overline{l}_i}) \leq b; -\log a \leq \overline{S}_{\overline{l}_i} + S_{t_k} \leq \log a\big] \\
&\leq \sum_{k,i=0}^{\infty} \mathbb{P}\big[b(\overline{R}_{\overline{l}_i}) \leq b; -\log a \leq \overline{S}_{\overline{l}_i} + S_{t_k} \leq \log a\big] \quad \text{since } b(L_{t_k}) \geq 0 \\
&= \sum_{i=0}^{\infty} \mathbb{E}\Big[\mathbf{1}_{[b(\overline{R}_{\overline{l}_i})\leq b]}\mathbb{E}\Big[\sum_{k=0}^{\infty} \mathbf{1}_{\{-\log a \leq \overline{S}_{\overline{l}_i} + S_{t_k} \leq \log a\}}\Big|\overline{g_i}, i \geq 0\Big]\Big] \\
&\leq C\log a \sum_{i=0}^{\infty} \mathbb{P}\big[b(\overline{R}_{\overline{l}_i}) \leq b\big]
\end{aligned}$$

Since we assume $B \geq 1$ a.s., we have for $i \geq 1$:

$$b(\overline{R}_{\overline{l}_i}) = b(\overline{R}_{\overline{l}_{i-1}} \overline{R}_{\overline{l}_i}^{\overline{l}_{i-1}}) = \mathrm{e}^{-\overline{S}_{\overline{l}_{i-1}}} b(\overline{R}_{\overline{l}_i}^{\overline{l}_{i-1}}) + b(\overline{R}_{\overline{l}_{i-1}}) \geq \mathrm{e}^{-\overline{S}_{\overline{l}_{i-1}}}$$

That is

$$U([-b,b]\times[1/a,a]) \leq C\log a\,(1 + \sum_{i=1}^{\infty} \mathbb{P}\big[\overline{S}_{\overline{l}_{i-1}} \geq -\log b\big] \leq C\log a\,(1 + C\log b)$$

Thus, if we observe that

$$V_z^{-1}V_z \subseteq \{(B,A) | |B| \leq 2b_0 a_0\, z,\ a_0^{-2} \leq A \leq a_0^2\}$$

then

$$U(V_z^{-1}V_z) \leq K \log a_0 (1 + \log z + \log(2b_0a_0)).$$

To estimate $U(V_l)$ from below as in the previous case, we just apply the renewal theorem (3.5). Plugging those estimates into (3.9), we conclude. $\square$

Proof of Proposition 3.1.

Step 1: Proof of (1) We apply Corollary 3.4 with $[k_1, k_2] = [-z, z]$ and $[m_1, m_2] = [-2b_0, 2b_0]$ and consider, according to the notation there, the subset of $\mathrm{Aff}(\mathbb{R})$

$$W(-2b_0, 2b_0, -z, z) = \big\{g \in \mathrm{Aff}(\mathbb{R}) | g([-z, z]) \subseteq [-2b_0, 2b_0]\big\} = \big\{(B, A)|\ Az + B < 2b_0\big\}.$$

This subset contains the set

$$V_z = \left\{(B, A)\Big|\frac{b_0^{-1}}{z} < A < \frac{b_0}{z},\ |B| < b_0\right\}.$$

We can apply Corollary 3.4 and, choosing b_0 large enough, Lemma 3.6.2 to conclude:

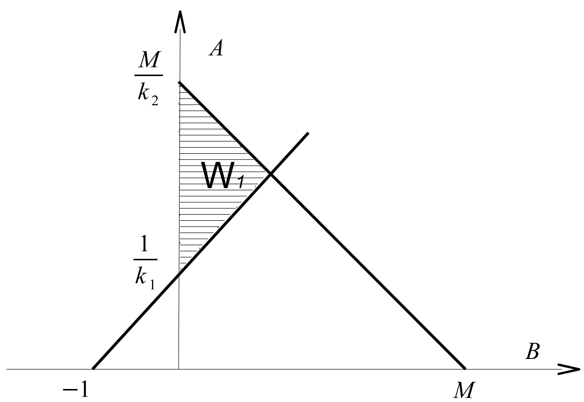
$$\nu(-z, z) \leq \frac{\nu[-2b_0, 2b_0]}{\mathbb{P}(T_{V_z} < \infty)} < C_0(1 + \log(z)).$$

Step 2: Proof of (2). Take $M > 1$ and $0 < k_1 < k_2$. Set $[m_1, m_2] = [z, zM]$. Then by Corollary 3.4

$$\nu[z, zM] \geq \mathbb{P}(T_{W_z} < \infty)\nu[k_1, k_2],$$

where

$$W_z = W(z, zM, k_1, k_2) = (0, z)W(1, M, k_1, k_2) =: (0, z)W_1.$$



Observe that if k_1, M and M/k_2 tend to infinity, then

$$W_1 = \{(B, A)|\ Ak_1 - B > 1, Ak_2 + B < M\}$$

grows to $\mathrm{Aff}(\mathbb{R})$. Thus, there exists $C > 0$ such if $k_1 \geq C, M > C$ and $M/k_2 \geq C$, the set W_1 contains the compact set V_0 defined in Lemma 3.6. Therefore $\mathbb{P}(T_{W_z} < \infty)$ is uniformly bounded from below for large values of z. Moreover, since we require the support of ν to be unbounded on the right, one can choose k_2 such that $\nu[k_1, k_2] > 0$ and we conclude.

Step 3: Proof (3) Let a_0, b_0 be sufficiently large numbers such that Lemma 3.6 holds and take $M > \max\{2, 4a_0^2\}$.

First suppose that $\frac{u_2}{u_1} < \frac{M}{4a_0^2}$. Take $[m_1, m_2] = [\mathrm{e}^x, \mathrm{e}^x M]$ and $[k_1, k_2] = [\mathrm{e}^{x+y}u_1, \mathrm{e}^{x+y}u_2]$. For $x > \log(b_0)$, the set

$$W(\mathrm{e}^x, \mathrm{e}^x M, \mathrm{e}^{x+y}u_1, \mathrm{e}^{x+y}u_2) = \{(B, A) \in \mathrm{Aff}(\mathbb{R}) | A\mathrm{e}^{x+y}u_2 + B \leq \mathrm{e}^x M; \quad A\mathrm{e}^{x+y}u_1 - B \geq \mathrm{e}^x\}$$

contains the set

$$V(y) = \Big\{(B, A) \in \mathrm{Aff}(\mathbb{R}) | B < b_0, \frac{2}{\mathrm{e}^y u_1} \leq A \leq \frac{M}{\mathrm{e}^y 2u_2}\Big\}.$$

Since $(\frac{M}{e^y 2u_2})/(\frac{2}{e^y u_1}) = \frac{Mu_1}{4u_2} > a_0^2$, we can apply Lemma 3.6 (and by Corollary 3.4) and prove that there exist $C >$ such that

$$\frac{\nu[e^{x+y}u_1, e^{x+y}u_2]}{\nu[e^x, e^x M]} \leq \frac{1}{\mathbb{P}(T_{V(y)} < \infty)} < C(1+y) \quad \text{for all } x > \log b_0, y > 0$$

By points **(1)** and **(2)**, the previous inequality is verified for $0 < x \leq \log b_0$ and all $y > 0$.

For general $U = [u_1, u_2]$ with $\frac{u_2}{u_1} \geq \frac{M}{4a_0^2}$ we can deduce (3.2) covering U with a finite number of small intervals.

□

Since we have supposed that the law of $\log A$ is aperiodic, proceeding as in [2] and [6], one can prove that the family of quotient measures is asymptotically invariant under the action of $\mathbb{R}_+^*$ and converges toward the Haar measure of $\mathbb{R}_+^*$.

Corollary 3.11. *Under hypotheses of Theorem 1.3. For all $\phi \in C_c(0, +\infty)$ not identically 0 and nonnegative, then $\liminf_{z\to\infty} \delta_{z^{-1}} * \nu(\phi) > 0$. Furthermore for $\phi_1, \phi_2 \in C_c(0, +\infty)$ and ϕ_2 not identically zero :*

$$\lim_{z\to\infty} \frac{\delta_{z^{-1}} * \nu(\phi_1)}{\delta_{z^{-1}} * \nu(\phi_2)} = \frac{\int_{\mathbb{R}_+^*} \phi_1(a)\frac{da}{a}}{\int_{\mathbb{R}_+^*} \phi_2(a)\frac{da}{a}}. \tag{3.12}$$

Thus one has

$$\lim_{x\to+\infty} \frac{\delta_{e^{-(x+y)}} * \nu(\phi)}{\delta_{e^{-x}} * \nu(\phi)} = 1 \tag{3.13}$$

and

$$\frac{\delta_{e^{-(x+y)}} * \nu(\phi)}{\delta_{e^{-x}} * \nu(\phi)} \leq K_\phi(1+y) \text{ for all } x, y > 0.$$

In particular the function $L(z) = \delta_{z^{-1}} * \nu(\phi)$ is slowly varying.

Proof. For the reader's convenience we give a sketchy proof (see Proposition 2.2 [6] for more details). First take a Lipschitz function Φ whose compact support contains $(1, M)$ and let $L(z) = \delta_{z^{-1}} * \nu(\Phi)$. We have proved that the family of measures $\frac{\delta_{z^{-1}} * \nu}{L(z)}$ is weakly compact. Then for every sequence we can extract a subsequence z_n along which this family of measures converges to a limit measure η.

For every Lipschitz compactly supported function ϕ and $\Psi \in \mathfrak{F}$ there exists a compact set $U = U(\phi, \Psi)$ such that

$$\left|\phi\Big(\frac{\Psi(u)}{z}\Big) - \phi\Big(\frac{Au}{z}\Big)\right| \leq \frac{B}{z} \cdot \mathbf{1}_U\Big(\frac{Au}{z}\Big),$$

and

$$\begin{aligned}
\lim_{n\to\infty} \frac{\left|\int \phi\Big(\frac{\Psi(u)}{z_n}\Big)\nu(du) - \int \phi\Big(\frac{Au}{z_n}\Big)\nu(du)\right|}{L(z_n)} &\leq \lim_{n\to\infty} \frac{C|z_n^{-1}b|\nu(a^{-1}z_n U)}{L(z_n)} \\
&\leq C\eta(a^{-1}U) \cdot \lim_{n\to\infty} |z_n^{-1}b| = 0,
\end{aligned}$$

Thus the function

$$h(y) = \delta_y * \eta(\phi) = \lim_{n\to\infty} \frac{\delta_{(0, z_n^{-1}y)} *_G \nu(\phi)}{L(z_n)}$$

on $\mathbb{R}_+^*$ is superharmonic with respect to the action of μ_A, the law A_1. Since h is positive and continuous, by the Choquet-Deny theorem, h is constant, that is $\delta_a * \eta(\phi) = \eta(\phi)$ for every $a \in \mathbb{R}_+^*$. Since $\eta(\Phi) = 1$, then η is a fixed multiple of the Haar measure of $\mathbb{R}_+^*$ and

$$\lim_{z\to+\infty} \frac{\delta_{z^{-1}} * \nu(\phi)}{\delta_{z^{-1}} * \nu(\Phi)} = \frac{\int \phi(a)\frac{da}{a}}{\int \Phi(a)\frac{da}{a}}.$$

This prove (3.12) and (3.13). In particular if ϕ is not null

$$\liminf_{z\to\infty} \delta_{z^{-1}} * \nu(\phi) \geq \int \phi(a)\frac{da}{a} \cdot \liminf_{z\to\infty} \delta_{z^{-1}} * \nu(\Phi) > 0$$

by Proposition 3.1.

Take k such that the support of ϕ is contained in $[1/k, k]$. Then

$$\frac{e^{-(x+y)} * \nu(\phi)}{e^{-x} * \nu(\phi)} \leq \frac{\nu[\mathrm{e}^x/M, \mathrm{e}^x M]}{e^{-x} * \nu(\phi)} \frac{\nu[\mathrm{e}^{x+y}/k, \mathrm{e}^{x+y}k]}{\nu[\mathrm{e}^x/M, \mathrm{e}^x M]} \leq K(1+y),$$

since the first quotient is bounded. □

4. Homogeneity at infinity

In this section we are going to finish the proof of Theorem 1.3. The main idea of the proof is similar to our previous papers [6, 8, 7]. Given a nice function ϕ on $\mathbb{R}_+^*$ we define the function

$$f(x) = \int_{\mathbb{R}_+^*} \phi(e^{-x}u)\nu(du).$$

Behavior at infinity of the measure ν is coded in the asymptotic behavior of f. To describe f we consider it as a solution of the Poisson equation

$$\overline{\mu} *_{\mathbb{R}} f(x) = f(x) + g(x)$$

where $\overline{\mu}$ is the law of $-\log A$ and the function g is defined by the equation above. We cannot use the classical renewal theorem, since the measure $\overline{\mu}$ is centered. In our previous papers we expressed f as a special potential of g, however this approach was technically involved and it was not possible to isolate the optimal hypotheses. Here we take advantage of our preliminary estimates of the measure ν, which are much stronger than in the previous papers and we apply ideas due to Durrett and Liggett [12], who studying similar equation and applying the duality lemma, were able to reduce the problem to the classical renewal equation.

The most technical part of the proof is contained in Lemma 4.1. We state there the assumptions that we need to require on the Poisson equation in order to control the asymptotic behavior of the solution. Unfortunately we are not able to justify that the functions f and g defined above satisfy hypotheses of the Lemma for all measures ν. The obstruction is not due to the asymptotic estimates at $+\infty$, but to the local behavior of the measure. We can prove integral conditions on f, only if we control local behavior of ν at 0. It turn out that translating the measure ν by some (appropriately chosen) vector v_0, the assumption of Lemma 4.1 will be satisfied. The details will be figure out in Lemma 4.7. Finally we deduce our main result.

Lemma 4.1. *Let $\overline{\mu}$ be a centered probability measure on $\mathbb{R}$ with moment of order $2+\epsilon$ for some $\epsilon > 0$ and f be a continuous function on $\mathbb{R}$ such that*

$$0 \leq f(x) \leq C(1+x^+) \qquad \textit{and} \qquad \int_{-\infty}^{y} f(x)dx \leq C(1+y^+) \tag{4.2}$$

where $x^+ := \max\{0, x\}$. Let g be the continuous function on $\mathbb{R}$ defined by the Poisson equation:

$$\overline{\mu} * f(x) = f(x) + g(x). \tag{4.3}$$

Suppose also that g is directly Riemann integrable, then

$$\lim_{x\to+\infty} \mathbb{E}[f(x+S_t)] - f(x) = \frac{-1}{\mathbb{E}[S_l]} \int_{\mathbb{R}} g(x)dx, \tag{4.4}$$

where S_n is the random walk of law $\overline{\mu}$ and t and l are the stopping times

$$t = \inf\{n > 0 : S_n \geq 0\} \text{ and } l = \inf\{n > 0 : S_n < 0\}.$$

Moreover, if $\int_{\mathbb{R}} g(x)dx = 0$ and $\int_{\mathbb{R}} |xg(x)|dx < \infty$,

$$\lim_{x\to+\infty} \mathbb{E}\left[\int_x^{x+S_t} f(z)dz\right] = \frac{1}{\mathbb{E}[S_l]} \int_{\mathbb{R}} g(x)xdx. \tag{4.5}$$

The proof of the this Lemma will be given in Appendix B.

Let ν be a μ-invariant Radon measure on $\mathbb{R}$. We would like to apply the previous lemma to the function $f(x) = \delta_{\mathrm{e}^{-x}} * \nu(\phi)$ for some fixed positive function $\phi \in C^1_C(\mathbb{R}^*_+)$, but it is not possible to do it directly since this function may not be sufficiently integrable at $-\infty$ (the reason is that we are not able to control local properties of ν and its behavior near 0). However there exists a slight perturbation of such a function that will satisfy to the conditions needed. Given $\phi \in C^1_C(\mathbb{R}^*_+)$ and $v_0 > 0$ define

$$\begin{aligned} f_\phi(x) &:= \int_{\mathbb{R}} \phi(e^{-x}(u - v_0))\nu(du), \\ g_\phi(x) &:= \overline{\mu} *_{\mathbb{R}} f_\phi(x) - f_\phi(x). \end{aligned}$$

Observe that $f_\phi(x) = \delta_{\mathrm{e}^{-x}} * \nu_0(\phi)$ where ν_0 is the measure ν translated by v_0:

$$\nu_0(\phi) = \int_{\mathbb{R}} \phi(u - v_0)\nu(du),$$

that is the invariant measure of the iterated functions system obtained by conjugating with the translation the original one:

$$\psi_0(x) = \psi(x + v_0) - v_0.$$

Denote by μ_0 its law. Observe that $A(\psi_0) = A(\psi)$ and we can choose $B(\psi_0) = Av_0 + v_0 + B$, thus μ_0 verifies to our main hypotheses if μ does. Since the translation doesn't change the asymptotic behavior, the two measures ν_0 and ν have the same behavior at $+\infty$, namely :

$$\lim_{x\to+\infty} f_\phi(x) - \delta_{\mathrm{e}^{-x}} * \nu(\phi) = 0. \tag{4.6}$$

In fact

$$\begin{aligned} \int_{-\infty}^{+\infty} |\phi(e^{-x}(u - v_0)) - \phi(e^{-x}u)|\nu(du) &\leq C \int_0^\infty |e^{-x}v_0| \mathbf{1}_{[\mathrm{e}^x m, \mathrm{e}^x(M+v_0)]}\nu(du) \\ &\leq C|e^{-x}v_0| \log(\mathrm{e}^x(M + v_0)) \end{aligned}$$

when $\mathrm{supp}(\phi) \subset [m, M]$. Summarizing, translation of the invariant measure does not change neither the problem we study nor our assumptions. Existence of a corresponding v_0 is provided by the following lemma, whose proof will be given in Appendix C.

Lemma 4.7. *There exists $v_0 > 0$ such that for all $\phi \in C^1_C(\mathbb{R}^*_+)$ the functions f_ϕ and g_ϕ satisfy the hypotheses of Lemma 4.1.*

Now we are ready to prove our main result.

Proof of Theorem 1.3. We claim that $\int g_\phi(y)dy = 0$. In fact for all y we can apply Corollary 3.11

$$\lim_{x\to+\infty}\frac{f_\phi(x+y)}{f_\phi(x)} = \lim_{x\to+\infty}\frac{\delta_{\mathrm{e}^{-(x+y)}} * \nu_0(\phi)}{\mathrm{e}^{-x} * \nu_0(\phi)} = 1;$$

thus, since $\mathbb{E}(S_t)$ is finite, by dominated convergence $\mathbb{E}(f_\phi(x+S_t)/f_\phi(x))$ also converges to 1. Fix $\varepsilon > 0$, then there exists x_ε such for all $x \geq x_\varepsilon$

$$\left|\mathbb{E}\big[f_\phi(x+S_t)\big] - f_\phi(x) + \frac{1}{\mathbb{E}S_t}\int g_\phi(y)dy\right| < \varepsilon \text{ and } \left|\frac{\mathbb{E}[f_\phi(x+S_t)]}{f_\phi(x)} - 1\right| < \varepsilon,$$

thus $f_\phi(x) \geq |\int g_\phi(y)dy|/(\varepsilon\mathbb{E}S_t) - 1$. Since by Lemma 4.7, $\int_{-\infty}^x f_\phi(y)dy < C(1+x)$, for all $x > x_\varepsilon > 0$

$$C(1+x) \geq \int_{x_\varepsilon}^x f_\phi(y)dy \geq \left(\frac{\left|\int g_\phi(y)dy\right|}{\varepsilon\mathbb{E}S_t} - 1\right)(x - x_\varepsilon).$$

That is

$$\left|\int_{\mathbb{R}} g_\phi(y)dy\right| \leq \varepsilon\mathbb{E}S_t\left(\liminf_{x\to+\infty}\frac{C(1+x)}{x-x_\varepsilon} + 1\right) = \varepsilon\mathbb{E}S_t(C+1).$$

Letting $\varepsilon \searrow 0$, we conclude.

In view of Corollary 3.11, the quotient $f_\phi(x+y)/f_\phi(x)$ is uniformly dominated by $1 + S_t$ for $x > 0$ and $0 < y < S_t$, thus

$$\lim_{x\to\infty}\int_0^{S_t}\frac{f_\phi(x+y)}{f_\phi(x)}dy = \int_0^{S_t} 1\ dy = S_t. \qquad \mathbb{P}\text{-almost surely}$$

By Fatou's lemma

$$\liminf_{x\to\infty}\mathbb{E}\left[\int_0^{S_t}\frac{f_\phi(x+y)}{f_\phi(x)}dy\right] \geq \mathbb{E}\left[\liminf_{x\to\infty}\int_0^{S_t}\frac{f_\phi(x+y)}{f_\phi(x)}dy\right] = \mathbb{E}[S_t] \tag{4.8}$$

Therefore by Lemma 4.1

$$\limsup_{x\to\infty} f_\phi(x) = \limsup_{x\to\infty}\frac{\mathbb{E}\left[\int_0^{S_t} f_\phi(x+y)dy\right]}{\mathbb{E}\left[\int_0^{S_t}\frac{f_\phi(x+y)}{f_\phi(x)}dy\right]} \leq \frac{1}{\mathbb{E}[S_l]\mathbb{E}[S_t]}\int_{\mathbb{R}} g_\phi(x)xdx$$

In particular this proves that $f_\phi(x)$ is bounded from above. Since by Corollary 3.11, we already know that $f_\phi(x)$ is bounded by below, $\int_0^{S_t}\frac{f_{\phi_i}(x+y)}{f_\phi(x)}dy < CS_t$. This allow to use in (4.8) dominated convergence theorem instead of Fatou's Lemma and to replace inferior limit with real limit and inequality with equality. Thus we have

$$\lim_{x\to\infty} f_\phi(x) = \lim_{x\to\infty}\frac{\mathbb{E}\left[\int_0^{S_t} f_\phi(x+y)dy\right]}{\mathbb{E}\left[\int_0^{S_t}\frac{f_\phi(x+y)}{f_\phi(x)}dy\right]} = \frac{1}{\mathbb{E}[S_l]\mathbb{E}[S_t]}\int_{\mathbb{R}} g_\phi(x)xdx = \frac{1}{\sigma^2}\int_{\mathbb{R}} g_\phi(x)xdx,$$

where $\sigma^2 = \int x^2\overline{\mu}(dx)$ (see [14] for the proof that $\mathbb{E}[S_l]\mathbb{E}[S_t] = \sigma^2$).

To conclude take a nonzero nonnegative function $\Phi \in C_c^1(0,+\infty)$, we have proved that the following limit exists

$$\lim_{z\to+\infty}\delta_{z^{-1}} * \nu(\Phi) = \lim_{x\to+\infty} f_\Phi(x) = C > 0$$

and by Corollary 3.11 is strictly positive. The same corollary also implies that for all $\phi \in C_c(0,+\infty)$

$$\lim_{z\to+\infty}\delta_{z^{-1}} * \nu(\phi) = \lim_{z\to+\infty}\frac{\delta_{z^{-1}} * \nu(\phi)}{\delta_{z^{-1}} * \nu(\Phi)}\lim_{z\to+\infty}\delta_{z^{-1}} * \nu(\Phi) = \frac{C}{\int_{\mathbb{R}}\Phi(a)\frac{da}{a}}\int_{\mathbb{R}}\phi(a)\frac{da}{a}.$$

□

5. Uniqueness of the invariant measure

Proof of Theorem 1.9. Notice first that for any compact set K

$$\begin{aligned}\lim_{n\to\infty} \mathbf{1}_K(X_n^y)|X_n^y - X_n^{y'}| &\leq |y-y'| \limsup_{n\to\infty} A_1 \dots A_n \mathbf{1}_K(X_n^y) \\ &= |y-y'| \limsup_{n\to\infty} \frac{X_n^y \mathbf{1}_K(X_n^y)}{\frac{X_n^y}{A_1\cdots A_n}} \\ &\leq \limsup_{n\to\infty} \frac{C(K)}{\frac{X_n^y}{A_1\cdots A_n}}.\end{aligned}$$

Thus it is sufficient to prove that

$$\liminf_{n\to\infty} \frac{X_n^y}{A_1\cdots A_n} = +\infty.$$

Notice that the sequence $\frac{X_n^y}{A_1\cdots A_n}$ in nondecreasing. Indeed, since $\Psi_n(X_{n-1}^y) \geq A_n X_{n-1}^y$,

$$\frac{X_n^y}{A_1\cdots A_n} = \frac{\Psi_n(X_{n-1}^y)}{A_1\cdots A_n} \geq \frac{X_{n-1}^y}{A_1\cdots A_{n-1}}.$$

Therefore it is enough to justify that for arbitrary large fixed $M>0$ the sequence is a.s. at least once greater than M. Let

$$U_{\beta,\gamma} := \{\Psi \in \mathfrak{F} | \Psi[0,+\infty) \subseteq [\beta,+\infty) \text{ and } A(\Psi) < \gamma\}$$

and

$$V_\alpha := \{\Psi \in \mathfrak{F} | A(\Psi) < \alpha\}.$$

By our hypotheses there exists $\alpha<1$, $\beta>0$, and γ such that these two sets have positive probability. For a fixed x_0, take $N>0$ such that $\alpha^{N-1} M\gamma x_0 < \beta$ and let $\psi_0 = \psi_1\psi_2$ with $\psi_1 \in U_{\beta,\gamma}$ and $\psi_2 \in V_\alpha^{N-1}$. We claim then that

$$\frac{\psi_0(x)}{A(\psi_0)x} > M \text{ for all } 0 \leq x \leq x_0. \tag{5.1}$$

In fact

$$\psi_0(x) = \psi_1(\psi_2(x)) \geq \beta > M(\gamma\alpha^{N-1}x_0) > MA(\psi_1)A(\psi_2)x > MA(\psi_0)x.$$

Observe that since X_n^y is recurrent there exists $x_0>1$ such that $\mathbb{P}[0 \leq X_n^y < x_0 \ i.o.] = 1$ for every $y \geq 0$. Let us fix y, x_0 and define a sequence T_k of hitting times of $[0,x_0]$

$$\begin{aligned}&T_0 = 0, \\ &T_k = \inf\left\{n > T_{k-1} + N : X_n^y < x_0\right\}.\end{aligned}$$

By recurrence, all T_k are almost surely finite. Let $\Psi_i^j := \Psi_j \circ \cdots \circ \Psi_{i+1}$, then $\{\Psi_{T_k}^{T_k+N}\}$ is a sequence of i.i.d. random transformations distributed as μ^N. Since $\mu^N(U_{\beta,\gamma}V_\alpha^{N-1}) > 0$ there exists almost surely k_0 such that $\Psi_{T_{k_0}}^{T_{k_0}+N} \in U_{\beta,\gamma}V_\alpha^{N-1}$. Then, by (5.1), we have:

$$\frac{X_{T_{k_0}+N}^y}{A_1\cdots A_{T_{k_0}+N}} = \frac{\Psi_{T_{k_0}}^{T_{k_0}+N}(X_{T_{k_0}}^y)}{A_1\cdots A_{T_{k_0}+N}} \geq \frac{\Psi_{T_{k_0}}^{T_{k_0}+N}(X_{T_{k_0}}^y)x_0}{A_{T_{k_0}+1}\cdots A_{T_{k_0}+N}X_{T_{k_0}}^y} > M.$$

□

6. Examples

In this section we present some of the more significant classes of stochastic dynamical system to which the results of the previous sections apply.

6.1. **The random difference equation.** The first example is naturally the SDS induced by random affinities, that is $\Psi_n(x) = A_n x + B_n$, for a random pair $(B_n, A_n) \in \mathbb{R}\times\mathbb{R}_+^*$. Then X_n^x is given by formula (1.7). This process is called the random difference equation or the affine recursion. It is well known that under the assumptions of Theorem 1.3 this process is recurrent and locally contractive, thus it possesses a unique invariant Radon measure ν, see [2]. Behavior of this measure at infinity was studied previously in [8, 6, 7] under a number of additional strong hypotheses. Theorem 1.3 provides an optimal result, in the sense that the hypotheses implying existence (and uniqueness) of the invariant measure, are sufficient also to deduce that this measure must behave at infinity like $\frac{Cdx}{x}$.

6.2. **Stochastic recursions with unique invariant measure.** Our results can also be applied to a more general class of stochastic recursions that behave at infinity as Ax (i.e. $\Phi(x) \sim Ax$ for large x). Those recursions in the contracting case ($\mathbb{E}[\log A] < 0$) were studied by Goldie [15] (see also Mirek [21], who described this class of recursions in general settings, including more examples). Just to give some concrete examples let us mention that our results are valid (under rather obvious and easy to formulate assumptions) for the following examples

- $\Psi_{1,n}(x) = \max\{A_n x, B_n\} + C_n$, for $A_n, B_n, C_n > 0$.
- $\Psi_{2,n}(x) = \sqrt{A_n^2 x^2 + B_n x + C_n}$, for $A_n, B_n, C_n > 0$ and $\Delta = B^2 - 4A^2 C \le 0$

Notice that in both cases above the mappings $\Psi_{i,n}$ are Lipschitz with the Lipschitz coefficient equal to A. This is obvious for the first example. For the second one, denote $x_0 = -\frac{B}{2A^2}$, $D = -\frac{\Delta}{4A^2}$. Observe that since $\Psi_{2,n}(x) = \sqrt{A^2(x-x_0)^2 + D}$, its derivative

$$\Psi'_{2,n}(x) = \frac{A^2(x-x_0)}{\sqrt{A^2(x-x_0)^2 + D}} = A\frac{1}{\sqrt{1 + \frac{D}{A^2(x-x_0)^2}}} \nearrow A$$

is an increasing function that tends to A. Hence, under appropriate moment assumptions, the SDS on $\mathbb{R}_+$ generated by the random functions defined above satisfy assumptions of both Theorem 1.3 and Theorem 1.9. Thus the corresponding random process possesses a unique invariant measure, which behaves at infinity like $\frac{Cdx}{x}$.

Observe that if we do not suppose $\Delta = B^2 - 4A^2C \le 0$, then $\Psi_{2,n}$ are still asymptotically linear functions to which Theorem 1.3 applies, but we cannot prove uniqueness.

6.3. **Random automorphisms of the interval $[0, 1]$.** Iterated function systems acting on the real line can be conjugated in order to be seen as random automorphisms of the interval $[0, 1]$ that fix the end points. Our key property (AL) is translated in this setting requiring that the automorphisms "reflect" at the same way in 0 and in 1, in the sense that the derivative in these two points has to be the same. The B term is then related to the term of order two at these end points (or order $2-\alpha$, if we conjugate a SDS that satisfy (AL^α)). More precisely

Corollary 6.1. *Consider a SDS on $[0, 1]$ defined by random functions $\phi \in C([0, 1])$ that fix 0 and 1, differentiable at the extremities of the interval and such that*

$$\phi'(0) = \phi'(1) =: a_\phi.$$

Let

$$\beta_1^0 = \inf_{u\in[0,1/2]}(1-\phi(u))>0,\quad \beta_2^0=\inf_{u\in[0,1/2]}\frac{\phi(u)}{u}>0,\quad \beta_3^0=\sup_{u\in[0,1/2]}\left|\frac{\phi(u)-a_\phi u}{u^2}\right|<\infty.$$

$$\beta_1^1 = \inf_{u\in[1/2,1]}\phi(u)>0,\ \beta_2^1=\inf_{u\in[1/2,1]}\frac{1-\phi(u)}{1-u}>0,\ \beta_3^1=\sup_{u\in[1/2,1]}\left|\frac{\phi(u)-1-a_\phi(u-1)}{(u-1)^2}\right|<\infty.$$

Suppose that $\mathbb{E}\left[|\log a_\phi|^{2+\varepsilon}\right]<\infty$, $\mathbb{E}\left[|\log\beta_k^i|^{2+\varepsilon}\right]<\infty$, *for some* $\varepsilon>0$ *and all* i,k, *and that* $\mathbb{E}[\log a_\phi]=0$, *then this SDS on* $[0,1]$ *is the conjugated to an asymptotically linear SDS on* $\mathbb{R}$ *that satisfy hypotheses of Theorem 1.3. Therefore there exists at least one invariant Radon measure* $\widetilde{\nu}$ *on* $(0,1)$ *and for every such a measure* $\widetilde{\nu}$, *that charges a neighborhood of 0, there exists a strictly positive constant* C *such that for all* $0<a<b<1$

$$\lim_{z\to+\infty}\widetilde{\nu}(a/z,b/z)=C\log b/a$$

Proof. Let

$$r(u)=-\frac{1}{u}+\frac{1}{1-u}.$$

be a diffeomorphism of $(0,1)$ onto $\mathbb{R}$. In the technical Lemma D.1, whose proof is postponed in the appendix, we prove that the conjugated function $\Psi_\phi=r\circ\phi\circ r^{-1}$ satisfy (AL) for $A(\Psi_\phi)=1/a_\phi$ and

$$B(\Psi_\phi)<C_r\left(\frac{(1+a_\phi+\beta_3^0)}{a_\phi\beta_2^0}+\frac{1}{\beta_1^0}+\frac{(1+a_\phi+\beta_3^1)}{a_\phi\beta_2^1}+\frac{1}{\beta_1^1}\right),$$

where C_r depends only on the function r.

Thus, under the hypotheses of the corollary, the conjugated SDS satisfies the hypotheses of our main theorem.

Let $\widetilde{\mu}$ be the law of ϕ and $\mu=r*\widetilde{\mu}*r^{-1}$ the law of the conjugated SDS on $\mathbb{R}$. Then ν is a μ-invariant Radon measure on $\mathbb{R}$ if and only if $\widetilde{\nu}=r^{-1}*\nu$ is a $\widetilde{\mu}$-invariant Radon measure on $(0,1)$. Then by Theorem 1.3 and since $|r(u)+1/u|<2$ for $0<u<1/2$,

$$\begin{aligned}\left|\widetilde{\nu}\Big(\frac{a}{z},\frac{b}{z}\Big)-\nu\Big(-\frac{z}{a},-\frac{z}{b}\Big)\right| &= \left|\nu\Big(r\Big(\frac{a}{z}\Big),r\Big(\frac{b}{z}\Big)\Big)-\nu\Big(-\frac{z}{a},-\frac{z}{b}\Big)\right|\\ &\le \nu\Big(-\frac{z}{a}-2,-\frac{z}{a}+2\Big)+\nu\Big(-\frac{z}{b}-2,-\frac{z}{b}+2\Big)\to 0\end{aligned}$$

for $z\to+\infty$. Thus

$$\lim_{z\to+\infty}\widetilde{\nu}\big(a/z,b/z\big)=\lim_{z\to+\infty}\nu\Big(-\frac{z}{a},-\frac{z}{b}\Big)=C\log b/a.$$

□

6.4. **Additive Markov processes and power functions.** When an asymptotically linear stochastic dynamical system is conjugated by a homeomorphism of the real line that behaves as the exponential at infinity it is transformed in a SDS, that is asymptotic to translations or, by the reversed conjugation, to power function.

More precisely consider a SDS generated by functions ϕ such that

$$|\phi(x)-x+\mathrm{sign}(x)u_\phi|\le v_\phi\mathrm{e}^{-|x|} \tag{6.2}$$

for some constants u_ϕ and v_ϕ. This class comprehends mapping of $[0,\infty)$ that are equal to translations outside a bounded set, that is Markov additive process as defined in Aldous [1, sections C11, C33]. Let s be a continuous bijection of $\mathbb{R}$ such that

$$s(x)=\mathrm{e}^x \text{ for } x>1\quad\text{ and }\quad s(x)=-\mathrm{e}^{-x}\text{ for } x<-1.$$

Then the SDS generated by $\psi_\phi(x) = s \circ \phi \circ s^{-1}$ satisfies to hypothesis (AL) with $A(\psi_\phi) = \mathrm{e}^{-u_\phi}$. Thus, under moment conditions that can be obtained with standard calculations, if $\mathbb{E}(u_\phi) = 0$ there exists some invariant measure and it behaves at infinity as the Lebesgue's measure dx, that is

$$\lim_{z\to+\infty} \widetilde{\nu}(\alpha + z, \beta + z) = C(\beta - \alpha),$$

for every measure of unbounded support, some constant $C > 0$ and all $\beta > \alpha$.

In a similar way a SDS generated by function ϕ such that

$$|x|^a \cdot \mathrm{sign}(x)\mathrm{e}^{-b_1 \log(|x|+2)^\alpha} \leq \phi(x) \leq |x|^a \cdot \mathrm{sign}(x)\mathrm{e}^{+b_1 \log(|x|+2)^\alpha},$$

for some α is associated to an α-asymptotically linear system by the reverse conjugation $\psi_\phi(x) = s^{-1} \circ \phi \circ s$ and $A(\psi_\phi) = a$. Thus, under moment hypotheses and if $\mathbb{E}(\log a) = 0$, for any invariant measure $\widetilde{\nu}$ whose support in unbounded on the right there exists a strictly positive constant C such that for all $1 < \alpha < \beta$

$$\lim_{z\to+\infty} \widetilde{\nu}(\alpha^z, \beta^z) = C \log \frac{\log\beta}{\log\alpha}$$

6.5. **Population of Galton-Watson tree with random reproduction law.** Consider the following model of reproduction of a population. Let $\{\rho_\omega | \omega \in \Omega\}$ be the set of probability measures on the set of natural numbers $\mathbb{N}$ and $\lambda(d\omega)$ be a probability law on Ω. At each generation a law of reproduction ρ_ω is chosen according to $\lambda(d\omega)$ and each individual j is replaced by r_j offsprings, r_j chosen according to the law ρ_ω and independently from the other individuals. To prevent the extinction of the population a random immigration i_ω it added to the population. More formally if the population consists of $x \in \mathbb{N}$ individuals, the population of the following generation is

$$\psi_{\omega,\mathbf{r}}(x) = i_\omega + \sum_{j=1}^{x} r_j$$

where the reproduction law $\omega \in \Omega$ is chosen according to $\lambda(d\omega)$, the $\mathbf{r} = \{r_j\}_j$ are i.i.d of law ρ_ω and i_ω is a random variable. If every generation is independent from the previous ones then the evolution of the population is a SDS on $\mathcal{R} = \mathbb{N}$ of law $\mu(d\psi) = \otimes\rho_\omega(d\mathbf{r})\lambda(d\omega)$. Under moment hypothesis of second order $\mathbb{E}r_1^2 < \infty$, the law of iterated logarithm proves that the $\psi_{\omega,\mathbf{r}}$ are μ-almost surely α-asymptotically linear with an error of order x^α for all $\alpha > 1/2$ and

$$A(\psi) = A_\omega = \int_{\mathbb{N}} r\rho_\omega(dr) = \text{avarage number of offspring per individual for } \rho_\omega\ .$$

Unlike the classical Galton-Watson process, in our context A_ω is not constant, but varies from one generation to another. The key parameter, that decides whether the system is recurrent and how, is then $\mathbb{E}(\log A_\omega) = \int \log A_\omega \lambda(d\omega)$. In order to be able to apply Theorem 1.3, we need to control the moment of the $B(\psi)$ and according to the following this following lemma this is possible if we suppose a moment of order 4 for the number of descendants. Our estimates are fairly rough and this hypothesis could be probably improved, but this go beyond the purpose of our paper.

Lemma 6.3. *Suppose* $\mathbb{E}(r_1^4) = \int_\Omega \int_{\mathbb{N}} r_1^4 \rho_\omega(dr)\lambda(d\omega) < \infty$. *Let* $\alpha > 3/4$ *and*

$$B(\psi) = B_{\omega,\mathbf{r}} = \sup_{x\in\mathbb{N}} \frac{|\psi_{\omega,\mathbf{r}}(x) - A_\omega x|}{x^\alpha + 1},$$

then there exists a finite constant C_α *that only depend on* α *such that*

$$\mathbb{E}((\log^+ B(\psi))^{2+\epsilon}) \leq C_\alpha \left(1 + \mathbb{E}((\log^+ i_\omega)^{2+\epsilon}) + \mathbb{E}(r_1^4)\right)$$

Proof. Observe that
$$\frac{|\psi_{\omega,\mathbf{r}}(x)-A_\omega x|}{x^\alpha+1}\le i_\omega+\frac{\big|\sum_{j=1}^x(r_j-A_\omega)\big|}{x^\alpha+1}$$
Thus
$$(\log^+ B(\psi))^{2+\epsilon}\le C\bigg((C+\log^+ i_\omega)^{2+\epsilon}+\sup_{x\in\mathbb{N}}\bigg(\log^+\frac{\big|\sum_{j=1}^x(r_j-A_\omega)\big|}{x^\alpha+1}\bigg)^{2+\epsilon}\bigg)$$
Observe that $y_j:=r_j-A_\omega$ are centered random variables. For a fixed reproduction law ω denote $\mathbb{P}_\omega$ the quenched probability. Since under $\mathbb{P}_\omega$ the variables y_j are independent, $\mathbb{E}_\omega(y_{j_1}y_{j_2}y_{j_3}y_{j_4})=0$ if there exists an index j_k that is different from all the others. Then standard calculus shows that
$$\begin{aligned}\mathbb{E}_\omega\bigg[\sum_{j=1}^x y_j\bigg]^4 &= \sum_{j_1,j_2,j_3,j_4=1}^x \mathbb{E}_\omega(y_{j_1}y_{j_2}y_{j_3}y_{j_4})\\ &= x\mathbb{E}_\omega(y_1^4)+3x(x-1)\big(\mathbb{E}_\omega[y_1^2]\big)^2\le 4x^2\mathbb{E}_\omega(y_1^4)\end{aligned}$$

Finally
$$\begin{aligned}\mathbb{E}\bigg(\sup_{x\in\mathbb{N}}\bigg(\log^+\frac{\big|\sum_{j=1}^x(r_j-A_\omega)\big|}{x^\alpha+1}\bigg)^{2+\epsilon}\bigg) &\le C\mathbb{E}\bigg(\sum_{x\in\mathbb{N}}\bigg(\frac{\big|\sum_{j=1}^x y_j\big|}{x^\alpha+1}\bigg)^4\bigg)\\ &= C\mathbb{E}\bigg(\sum_{x\in\mathbb{N}}\frac{\mathbb{E}_\omega\big[\big|\sum_{j=1}^x y_j\big|^4\big]}{(x^\alpha+1)^4}\bigg)\\ &\le C\mathbb{E}\bigg(\sum_{x\in\mathbb{N}}\frac{4x^2\mathbb{E}_\omega(y_1^4)}{(x^\alpha+1)^4}\bigg)=C\sum_{x\in\mathbb{N}}\frac{4x^2\mathbb{E}(y_1^4)}{(x^\alpha+1)^4}<\infty\end{aligned}$$

since $\alpha>3/4$. □

6.6. **Reflected random walk in critical case.** The reflected random walk
$$Y_n^x=\big|Y_{n-1}^x-u_n\big|,$$
is an example of asymptotic translation for which (6.2) holds, thus we can apply our main Theorem 1.3. However, in this case the same results hold under weaker hypotheses and a much more direct proof. We give here the proof of Theorem 1.11, stated in the introduction.

Proof of Theorem 1.11. Define the upward ladder times of $S_n=\sum_{i=1}^n u_i$:
$$\begin{aligned}t_0&=0,\\ t_{k+1}&=\inf\{n>t_k:\ S_n\ge S_{t_k}\},\end{aligned}$$
and put $\overline{u}_k=S_{t_k}-S_{t_{k-1}}$, then $\{\overline{u}_k\}$ is a sequence of i.i.d. random variables and every $\overline{u}_k$ is equal in distribution to S_{t_1}. We define reflected random walk for $\{\overline{u}_k\}$:
$$\begin{aligned}\overline{Y}_0^x&=x,\\ \overline{Y}_{k+1}^x&=\big|\overline{Y}_k^x-\overline{u}_{k+1}\big|,\end{aligned}$$
then $\overline{Y}_k^x=Y_{t_k}^x$. In view of the result of Chow and Lai [9], $\mathbb{E}(\overline{u}_k)^{\frac12}<\infty$ and this is sufficient for existence of a unique invariant probability measure ν_L of the process $\{\overline{Y}_k^x\}$ (see [22] for more details). Then the measure
$$\nu_0(f)=\int_{\mathbb{R}_+}\mathbb{E}\bigg[\sum_{n=0}^{t-1}f(Y_n^x)\bigg]\nu_t(dx)$$

is μ invariant. Indeed, since $\mu_t * \nu_t = \nu_t$, we have

$$\begin{aligned}\mu * \nu_0(f) &= \int_{\mathbb{R}_+}\int_{\mathbb{R}} \mathbb{E}\Big[\sum_{n=0}^{t-1} f(Y_n^{|x-y|})\Big]\mu(dy)\nu_t(dx) \\ &= \int_{\mathbb{R}_+} \mathbb{E}\Big[\sum_{n=1}^{t} f(Y_n^{x})\Big]\nu_t(dx) = \nu_0(f).\end{aligned}$$

Define $l_i = \inf\{n > l_{i-1} : \ S_n < S_{l_{i-1}}\}$. Then, since $\mathbb{E}(u_1^-)^2 < \infty$, $-\infty < \mathbb{E}S_l < 0$ (see [9]). Then by the duality Lemma [14]

$$\nu_0(f) = \int_{\mathbb{R}_+} \mathbb{E}\Big[\sum_{n=0}^{t-1} f(x - S_n)\Big]\nu_t(dx) = \int_{\mathbb{R}_+} \mathbb{E}\Big[\sum_{n=0}^{\infty} f(x - S_{l_n})\Big]\nu_t(dx)$$

Take any $\phi \in C_C(\mathbb{R}_+)$. Then by the renewal theorem

$$\lim_{z\to\infty}\int_{\mathbb{R}_+} \phi(u-z)\nu_0(du) = \lim_{z\to\infty}\int_{\mathbb{R}_+} \mathbb{E}\Big[\sum_{n=0}^{\infty}\phi(x - S_{l_n} - z)\Big]\nu_t(dx) = \frac{1}{-\mathbb{E}S_l}\int_{\mathbb{R}_+}\phi(x)dx.$$

To justify that we may use the Lebesgue theorem take α and β such that supp$\phi \subset [\alpha, \beta]$ and notice that by the renewal theorem

$$\mathbb{E}\Big[\sum_{n=0}^{\infty}\phi(x - S_{l_n} - z)\Big] \leq C\mathbb{E}\big[\#n : \ \alpha < x - S_{l_n} - z < \beta\big] \leq C|\beta - \alpha|.$$

The same argument proves that ν_0 is a Radon measure, so $\nu_0 = C\nu$ and the Theorem is proved. $\square$

Appendix A. Proof of Lemmas 2.1

Proof. We will prove the result only for positive x, since for negative values of x the same argument is valid just by conjugating with the map $x \mapsto x$.

Suppose first $x \geq 1$. Then

$$\begin{aligned} r\left(A_\alpha r^{-1}(x) - B_\alpha(1 + |r^{-1}(x)|^\alpha)\right) &\leq \psi_r(x) \leq \ r\left(A_\alpha r^{-1}(x) + B_\alpha(1 + |r^{-1}(x)|^\alpha)\right) \\ r\left(A_\alpha x^{\frac{1}{1-\alpha}} - B_\alpha(1 + x^{\frac{\alpha}{1-\alpha}})\right) &\leq \psi_r(x) \leq \ r\left(A_\alpha x^{\frac{1}{1-\alpha}} + B_\alpha(1 + x^{\frac{\alpha}{1-\alpha}})\right) \\ r\left(A_\alpha x^{\frac{1}{1-\alpha}} - B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}}\right) &\leq \psi_r(x) \leq \ r\left(A_\alpha x^{\frac{1}{1-\alpha}} + B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}}\right)\end{aligned}$$

where c_α only depends on α. Suppose further $x > c_\alpha B_\alpha / A_\alpha$, then $A_\alpha x^{\frac{1}{1-\alpha}} - B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}} > 0$ and

$$\begin{aligned}\left(A_\alpha x^{\frac{1}{1-\alpha}} - B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}}\right)^{1-\alpha} &\leq \psi_r(x) \leq \ \left(A_\alpha x^{\frac{1}{1-\alpha}} + B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}}\right)^{1-\alpha} \\ A_\alpha^{1-\alpha} x^{\frac{1-\alpha}{1-\alpha}} - A_\alpha^{-\alpha} x^{\frac{-\alpha}{1-\alpha}} B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}} &\leq \psi_r(x) \leq \ A_\alpha^{1-\alpha} x^{\frac{1-\alpha}{1-\alpha}} + (1-\alpha)A_\alpha^{-\alpha} x^{\frac{-\alpha}{1-\alpha}} B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}} \\ A_\alpha^{1-\alpha} x - A_\alpha^{-\alpha} B_\alpha c_\alpha &\leq \psi_r(x) \leq \ A_\alpha^{1-\alpha} x + (1-\alpha)A_\alpha^{-\alpha} B_\alpha c_\alpha\end{aligned}$$

since for $x_0 > 0$ and $h > 0$, by concavity $(x_0 + h)^{1-\alpha} \leq x_0^{1-\alpha} + (1-\alpha)x_0^{-\alpha}h$ and $(x_0 - h)^{1-\alpha} \geq x_0^{1-\alpha} - x_0^{-\alpha}h$. Thus, we proved the lemma for $x > \max\{1, c_\alpha B_\alpha/A_\alpha\}$ Now, for $x < 1$

$$\begin{aligned} r\left(A_\alpha x^{\frac{1}{1-\alpha}} - B_\alpha(1 + x^{\frac{\alpha}{1-\alpha}})\right) &\leq \psi_r(x) \leq \ r\left(A_\alpha x^{\frac{1}{1-\alpha}} + B_\alpha(1 + x^{\frac{\alpha}{1-\alpha}})\right) \\ -\left(2B_\alpha\right)^{1-\alpha} &\leq \psi_r(x) \leq \ \left(A_\alpha + 2B_\alpha\right)^{1-\alpha}\end{aligned}$$

and for $x \le c_\alpha B_\alpha / A_\alpha$.

$$-\left(B_\alpha\left(1+\left(\frac{c_\alpha B_\alpha}{A_\alpha}\right)^{\frac{\alpha}{1-\alpha}}\right)\right)^{1-\alpha} \le \psi_r(x) \le \left(A_\alpha\left(\frac{c_\alpha B_\alpha}{A_\alpha}\right)^{\frac{1}{1-\alpha}} + B_\alpha\left(1+\left(\frac{c_\alpha B_\alpha}{A_\alpha}\right)^{\frac{\alpha}{1-\alpha}}\right)\right)^{\frac{\alpha}{1-\alpha}}$$

Hence the lemma follows. □

Appendix B. Proof of Lemma 4.1

Proof of Lemma 4.1. **Step 1.** Let t_k and l_k be the stopping times defined in (2.7). Let U_l be the potential of the random walk S_{l_k} and let

$$R(x) := \sum_{k=0}^{\infty} \mathbb{E}\big[g(x+S_{l_k})\big] = U_l(\delta_x *_{\mathbb{R}} g).$$

Since the function g is directly Riemann integrable and $-\infty < \mathbb{E}S_l < 0$, the function R is well defined and finite for every x. Notice also that by the duality lemma [14]

$$R(x) = \sum_{k=0}^{\infty} \mathbb{E}\big[g(x+S_{l_k})\big] = \mathbb{E}\Big[\sum_{k=0}^{t-1} g(x+S_k)\Big]. \tag{B.1}$$

Step 2. We claim that

$$\mathbb{E}\big[f(x+S_t)\big] - f(x) = \mathbb{E}\left[\sum_{k=0}^{t-1} g(x+S_k)\right] = R(x). \tag{B.2}$$

To prove this fact, observe that the process $f(x+S_n) - \sum_{k=0}^{n-1} g(x+S_k)$ forms a martingale (for this purpose one has just to iterate the Poisson equation (4.3)). Then for any fixed n, $T \wedge n$ is a bounded stopping time, therefore by the optional stopping time theorem we have

$$f(x) = \mathbb{E}\Big[f(x+S_{t\wedge n})\Big] - \mathbb{E}\left[\sum_{k=0}^{(t\wedge n)-1} g(x+S_k)\right].$$

To justify that we can let n tend to infinity and change the order of the limit and the expected value to obtain (B.2) observe that

$$\mathbb{E}\big[f(x+S_{t\wedge n})\big] \le C\mathbb{E}\big[1+(x+S_{t\wedge n})^+\big] \le C\mathbb{E}\big[1+(x+S_t)^+\big] < \infty.$$

While the second term is uniformly dominated in n by

$$\mathbb{E}\Big[\sum_{k=0}^{t-1} |g|(x+S_k)\Big] = \sum_{k=0}^{\infty} \mathbb{E}\big[|g|(x+S_{l_k})\big] < \infty,$$

thus converges to $R(x)$ when n goes to infinity.

This proves that

$$\mathbb{E}\big[f(x+S_t)\big] - f(x) = R(x) = U_l(\delta_x *_{\mathbb{R}} g)$$

and by the renewal theorem we obtain (4.4).

Step 3. Let

$$G(x) := \int_{-\infty}^{x} g(z)dz.$$

If we suppose $\int g(x)dx = 0$ then

$$G(x) = \int_{-\infty}^{+\infty} g(z)dz - \int_{x}^{\infty} g(z)dz = -\int_{x}^{+\infty} g(z)dz.$$

Thus

$$|G(x)| \leq \int_{-\infty}^{x} |g(z)|dz\, \mathbf{1}_{(-\infty,0]}(x) + \int_{x}^{\infty} |g(z)|dz\, \mathbf{1}_{[0,+\infty)}(x) =: G_1(x) + G_2(x),$$

and G is directly Riemann integrable since functions G_i are monotone and integrable on their support:

$$\begin{aligned}
\int_{-\infty}^{0} G_1(x)dx &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{1}_{[z<x<0]}|g(z)|dx\,dz = \int_{-\infty}^{0} |zg(z)|\,dz < \infty \\
\int_{0}^{+\infty} G_2(x)dx &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{1}_{[z>x>0]}|g(z)|dx\,dz = \int_{0}^{+\infty} |zg(z)|\,dz < \infty.
\end{aligned}$$

Furthermore

$$\begin{aligned}
\int_{-\infty}^{\infty} G(x)dx &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{1}_{[z<x<0]}g(z)dx\,dz - \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{1}_{[z>x>0]}g(z)dx\,dz \\
&= -\int_{-\infty}^{+\infty} zg(z)dz.
\end{aligned}$$

Step 4. By the renewal theory, $U_l(\delta_x *_{\mathbb{R}} G)$ is well defined and by Fubini's thoerem

$$\int_{-\infty}^{x} R(z)dz = \int_{-\infty}^{0}\int_{-\infty}^{x} g(z+u)dzU_l(du) = \int_{-\infty}^{0}\int_{-\infty}^{x+u} g(z)dzU_l(du) = U_l(\delta_x *_{\mathbb{R}} G).$$

On the other hand

$$\int_{-\infty}^{x} R(z)dz = \mathbb{E}\left[\int_{-\infty}^{x} f(z+S_t)dz - \int_{-\infty}^{x} f(z)dz\right] = \mathbb{E}\left[\int_{x}^{x+S_t} f(z)dz\right].$$

In fact the two integrals above are finite because by our hypotheses

$$\int_{-\infty}^{x} \mathbb{E}[f(y+S_t)]dy = \mathbb{E}\left[\int_{-\infty}^{x+S_t} f(y)dy\right] \leq C\mathbb{E}[1+(x+S_t)^+]$$

and $\mathbb{E}S_t < \infty$ since $\overline{\mu}$ has moment of order $2+\epsilon$, see [9]. Thus we proved

$$\mathbb{E}\left[\int_{x}^{x+S_t} f(z)dz\right] = \delta_x *_{\mathbb{R}} U_l(G)$$

and we can conclude using again the renewal theorem. □

Appendix C. Proof of Lemma 4.7

Proof. **Step 1.** Let $0<\gamma<1$, then the set of $v>0$ such that the function $u \mapsto (u-v)^{-\gamma}$ is ν-integrable on $(v,+\infty)$ is of full Lebesgue measure. In fact for all interval $[a,b] \subset (0,\infty)$:

$$\begin{aligned}
\int_a^b \left(\int_v^{\infty} (u-v)^{-\gamma}\nu(du)\right) dv &= \int_a^{2b}\left(\int_a^{u\wedge b} (u-v)^{-\gamma}dv\right)\nu(du) + \int_{2b}^{\infty}\left(\int_a^{u\wedge b}(u-v)^{-\gamma}dv\right)\nu(du) \\
&\leq \int_a^{2b}\left(\int_0^{2b-a} w^{-\gamma}dw\right)\nu(du) + \int_{2b}^{\infty}\left(\int_{u-b}^{u-a} w^{-\gamma}dw\right)\nu(du) \\
&= C + \int_{2b}^{\infty}(u-b)^{-\gamma}(b-a)\nu(du) < \infty
\end{aligned}$$

Take v_0 such that $\int_{v_0}^{\infty}(u-v_0)^{-\gamma}\nu(du)<\infty$ then

$$f_\phi(x)=\int_{v_0}^{\infty}\phi(e^{-x}(u-v_0))\nu(du)\le C\int_{v_0}^{\infty}e^{\gamma x}(u-v_0)^{-\gamma}\nu(du)\le Ce^{\gamma x},$$

this gives good estimates of f_ϕ for negative x's.

Step 2. Let $\mathrm{supp}(\phi)\subset[m,M]$ then, since by Proposition 3.1 the tail of ν is at most logarithmic, for $x\ge 0$,

$$f_\phi(x)\le\nu([e^x m+v_0,e^x M+v_0])\le\nu([e^x m,e^x(M+v_0)])\le C(1+x).$$

and

$$\begin{aligned}\int_{-\infty}^{x}f_\phi(y)dy &\le C\int_{\mathbb{R}}\int_{-\infty}^{\infty}\mathbf{1}_{[y<x]}\mathbf{1}_{[m,M]}(\mathrm{e}^{-y}(u-v_0))dy\nu(du)\\ &\le C\int_{\mathbb{R}}\mathbf{1}_{[v_0<u\le \mathrm{e}^x(M+v_0)]}\log\frac{M}{m}\nu(du)\le C(1+x^+)\end{aligned}$$

This proves (4.2).

Step 3. We need to show that $g_\phi=\overline{\mu}*f_\phi-f_\phi$ is directly Riemann integrable and that $\int_{\mathbb{R}}|xg(x)|dx<\infty$. For $x<0$:

$$\begin{aligned}\overline{\mu}*f_\phi(x) &= \int_{-\infty}^{+\infty}f_\phi(x+y)\overline{\mu}(dy)\\ &= \int_{-\infty}^{-x/2}C\mathrm{e}^{\gamma(x+y)}\overline{\mu}(dy)+\int_{-x/2}^{+\infty}K(1+(x+y)^+)\overline{\mu}(dy)\\ &\le C\mathrm{e}^{\gamma(x/2)}+\int_{-x/2}^{\infty}K(1+|y|)\overline{\mu}(dy)=C\mathrm{e}^{\gamma(x/2)}+\frac{1}{|x|^{2+\varepsilon}}\int_{-x/2}^{\infty}K(1+|y|)|y|^{1+\varepsilon}\overline{\mu}(dy)\\ &\le \frac{C}{1+|x|^{1+\varepsilon}},\end{aligned}$$

since $\overline{\mu}$ has a moment of order $2+\varepsilon$. Thus $g_\phi\mathbf{1}_{(-\infty,0]}$ is directly Riemann integrable. Furthermore

$$\begin{aligned}\int_{-\infty}^{0}|x|\overline{\mu}*f_\phi(x)dx &= \int_{-\infty}^{+\infty}\int_{-\infty}^{0}|x|f_\phi(x+y)dx\overline{\mu}(dy)\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{y}|x-y|f_\phi(x)dx\overline{\mu}(dy)\\ &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{0}|x-y|f_\phi(x)dx+\int_{0}^{y^+}|x-y|f_\phi(x)dx\right)\overline{\mu}(dy)\\ &\le \int_{-\infty}^{+\infty}\left(C\int_{-\infty}^{0}|x-y|\mathrm{e}^{\gamma x}dx+2|y|\int_{0}^{y^+}f_\phi(x)dx\right)\overline{\mu}(dy)\\ &\le C\int_{-\infty}^{+\infty}(1+|y|+|y|^2)\overline{\mu}(dy)<\infty\end{aligned}$$

Step 4. To check that g_ϕ is directly Riemann integrable and $|xg_\phi(x)|$ is integrable for positive x we show that:

$$\sum_{n=0}^{\infty}\sup_{n\le x<n+1}|xg_\phi(x)|<\infty.$$

Applying μ_0 invariance of ν_0 and since $A(\psi_0) = A(\psi)$, we obtain

$$|g(x)| = \Big|\int\int \phi\big(e^{-x}(A(\psi)u)\big) - \phi\big(e^{-x}(\psi(u))\big)\nu_0(du)\mu_0(d\psi)\Big|$$

Observe that $\widetilde{\phi}(x) = \phi(\mathrm{e}^x)$ is a Lipschitz function on $\mathbb{R}$ thus :

$$\begin{aligned}\Big|\phi\big(e^{-x}(A(\psi)u)\big) - \phi\big(e^{-x}\psi(u)\big)\Big| &\leq \min\big\{C\big|\log\frac{\psi(u)}{A(\psi)u}\big|, 2\|\phi\|_\infty\big\}\\ &\leq \min\big\{C\big|\frac{\psi(u)}{A(\psi)u} - 1\big|, 2\|\phi\|_\infty\big\}\\ &\leq \min\big\{C\big|\frac{B(\psi)}{A(\psi)u}\big|, 2\|\phi\|_\infty\big\} =: \rho(\frac{Au}{B})\end{aligned}$$

where we use the convention that $\log z = -\infty$ for $z \leq 0$ and $\rho(y) := \min\{C\big|\frac{1}{y}\big|, 2\|\phi\|_\infty\}$. Take now $0 \leq n \leq x < n+1$

$$\begin{aligned}&|x|\Big|\phi\big(e^{-x}(Au)\big) - \phi\big(e^{-x}\psi(u)\big)\Big|\\ &\qquad\leq \log^+\frac{Au+B}{m}\cdot\rho(\frac{Au}{B})\left(\mathbf{1}_{[\log\frac{\psi(u)}{M\mathrm{e}}\leq n\leq\log\frac{\psi(u)}{m}]} + \mathbf{1}_{[\log\frac{Au}{M\mathrm{e}}\leq n\leq\log\frac{Au}{m}]}\right).\end{aligned}$$

Thus

$$\begin{aligned}\sum_{n=0}^{\infty}\sup_{n\leq x<n+1}|xg_\phi(x)| &\leq \int\int\sum_{n=0}^{\infty}\sup_{n\leq x<n+1}|x|\Big|\phi\big(e^{-x}(Au)\big) - \phi\big(e^{-x}\psi(u)\big)\Big|\nu_0(du)\mu_0(d\psi)\\ &\leq \int\int\log^+\frac{Au+B}{m}\cdot\rho(\frac{Au}{B})2\log\frac{\mathrm{e}M}{m}\nu_0(du)\mu_0(d\psi)\\ &\leq 2\log\frac{\mathrm{e}M}{m}\int\Big(\int\Big(\log^+\frac{1}{m} + \log^+ B + \log^+(\frac{Au}{B}+1)\Big)\rho(\frac{Au}{B})\nu_0(du)\Big)\mu_0(d\psi)\end{aligned}$$

To estimate the last expression we will use the fact that there exists a constant C such that for all non-increasing functions $h : [0,+\infty) \to [0,+\infty)$ and all $M > 0$

$$\int_{\mathbb{R}} h\big(|u|/M\big)\nu_0(du) \leq C(1+\log^+ M)\left(\|h\|_\infty + \int_1^{+\infty} h(z)(1+\log(z))\frac{dz}{z}\right). \tag{C.1}$$

Before we prove the last inequality, let us check how it implies the lemma. Since $\log^+(z+1)\rho(z) \leq C/(1+z)^{1/2}$ for $z > 0$, by (C.1), we have

$$\begin{aligned}&\int\left(\log^+\frac{1}{m} + \log^+ B + \log^+(\frac{Au}{B}+1)\right)\rho(\frac{Au}{B})\nu_0(du)\\ &\leq C(1+\log^+(B/A))\left((1+\log^+ B) + \int_1^{+\infty}\left((1+\log^+ B)\rho(z) + \frac{1}{(1+z)^{1/2}}\right)(1+\log^+(z))\frac{dz}{z}\right)\\ &\qquad\leq C\left(1 + (\log^+ B)^2 + \log^+ B\log^+ A\right).\end{aligned}$$

The last expression is μ_0-integrable and we conclude.

Finally to prove (C.1) we write

$$\begin{aligned}\int_{\mathbb{R}} h\big(|u|/M\big)\nu_0(du) &\leq \|h\|_\infty \nu_0([-M\mathrm{e}, M\mathrm{e}]) + \int \mathbf{1}_{[|u|>\mathrm{e}M]} h\big(|u|/M\big)\nu_0(du) \\ &\leq C(1+\log^+ M)\|h\|_\infty + \sum_{n=1}^{\infty} \int \mathbf{1}_{[\mathrm{e}^{n+1}M \geq |u| > \mathrm{e}^n M]} \nu_0(du) h(e^n) \\ &\leq C(1+\log^+ M)\|h\|_\infty + \sum_{n=1}^{\infty} \big(\log^+(\mathrm{e}^{n+1}M)+1\big) h(e^n) \\ &\leq C(1+\log^+ M)\|h\|_\infty + \sum_{n=1}^{\infty} \int_{\mathrm{e}^{n-1}}^{\mathrm{e}^n} \big(\log^+(z\mathrm{e}^2 M)+1\big) h(z)\frac{dz}{z} \\ &\leq C(1+\log^+ M)\|h\|_\infty + \int_1^{\infty} \big(\log^+(z)+\log^+ M+3\big) h(z)\frac{dz}{z}.\end{aligned}$$

□

Appendix D. SDS of the interval conjugate to (AL) SDS

Lemma D.1. *Let $\phi \in C([0,1])$ be a function fixing 0 and 1, derivable at 0 and 1 and such that $\phi'(0) = \phi'(1) =: a_\phi$. Suppose:*

$$\begin{aligned}\beta_1^0 &= \inf_{u\in[0,1/2]}(1-\phi(u)) > 0, \quad \beta_2^0 = \inf_{u\in[0,1/2]} \frac{\phi(u)}{u} > 0, \quad \beta_3^0 = \sup_{u\in[0,1/2]} \left|\frac{\phi(u)-a_\phi u}{u^2}\right| < \infty. \\ \beta_1^1 &= \inf_{u\in[1/2,1]} \phi(u) > 0,\ \beta_2^1 = \inf_{u\in[1/2,1]} \frac{1-\phi(u)}{1-u} > 0,\ \beta_3^1 = \sup_{u\in[1/2,1]} \left|\frac{\phi(u)-1-a_\phi(u-1)}{(u-1)^2}\right| < \infty.\end{aligned}$$

Consider the diffeomorphism of $(0,1)$ on $\mathbb{R}$

$$r(u) = -\frac{1}{u} + \frac{1}{1-u}.$$

Then $\Psi_\phi = r\circ\phi\circ r^{-1}$ satisfy (AL) for $A(\Psi_\phi) = 1/a_\phi$ and

$$B(\Psi_\phi) < C_r\left(\frac{(1+a_\phi+\beta_3^0)}{a_\phi\beta_2^0} + \frac{1}{\beta_1^0} + \frac{(1+a_\phi+\beta_3^1)}{a_\phi\beta_2^1} + \frac{1}{\beta_1^1}\right),$$

where C_r depends only on the function r.

Proof. Since the function r satisfies $r(u) = -r(1-u)$ and our assumptions on ϕ near 0 and 1 are symmetric it is sufficient to prove the condition (AL) only for negative x. Since $\beta_3^0 < \infty$, by the Taylor expansion we have

$$\phi(u) = au + \epsilon_\phi(u) \text{ with } |\epsilon_\phi(u)| \leq \beta_3^0 u^2 \text{ for } u \leq 1/2. \tag{D.2}$$

Moreover simple calculus shows that

$$r^{-1}(x) = -\frac{1}{x} + \epsilon_{r^{-1}}(x) \text{ with } \epsilon_{r^{-1}}(x) = O\left(\frac{1}{x^2}\right) \text{ for } x \to -\infty \tag{D.3}$$

For $x < 0$ we write

$$\left|\frac{x}{a_\phi} - \Psi_\phi(x)\right| = \left|\frac{x}{a_\phi} - r\big(\phi\big(r^{-1}(x)\big)\big)\right| \leq \left|\frac{x}{a_\phi} + \frac{1}{\phi\big(r^{-1}(x)\big)}\right| + \frac{1}{1-\phi\big(r^{-1}(x)\big)}$$

Notice that for $x<0$, $r^{-1}(x)\in(0,1/2)$, therefore the second factor can be bounded by $\frac{1}{\beta_1^0}$. So, we need just to estimate the first term. We write

$$I(x)=\left|\frac{x}{a_\phi}-\frac{1}{\phi\big(r^{-1}(x)\big)}\right|=\frac{|\phi(r^{-1}(x))x-a_\phi|}{|a_\phi\cdot\phi(r^{-1}(x))|}$$

Take $M=-r(1/10)$, then for $x\in[-M,0]$ we have $\phi(r^{-1}(x))\geq\beta_2^0r^{-1}(x)\geq\beta_2^0/10$ and hence

$$I(x)\leq 10\frac{M+a_\phi}{a_\phi\beta_2^0}.$$

Now we consider $x<-M$. Since there exists η such that $xr^{-1}(x)>\eta$, by (D.2) and (D.3), we have

$$\begin{aligned}
I(x) &= \frac{\big|\phi(r^{-1}(x))x+a_\phi\big|\cdot|x|}{a_\phi\cdot\frac{\phi(r^{-1}(x))}{r^{-1}(x)}\cdot|xr^{-1}(x)|}\leq\frac{1}{a_\phi\beta_2^0\eta}\cdot\big|\phi(r^{-1}(x))x+a_\phi\big||x|\\
&= \frac{1}{a_\phi\beta_2^0\eta}\big|a_\phi r^{-1}(x)\,x+\varepsilon_\phi(r^{-1}(x))\;x+a_\phi\big||x|=\frac{1}{a_\phi\beta_2^0\eta}\big|a_\phi\varepsilon_{r^{-1}}(x)x+\varepsilon_\phi\left(r^{-1}(x)\right)x\big|\\
&\leq \frac{\big|a_\phi\varepsilon_{r^{-1}}(x)x\big|+\beta_3^0\big|\left(r^{-1}(x)\right)^2x\big|}{a_\phi\beta_2^0\eta}\leq\frac{C_r(a_\phi+\beta_3^0)}{a_\phi\beta_2^0}.
\end{aligned}$$

□

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